Published by Institute of Physics Publishing for SISSA
Received: *June 10, 2008*
Accepted: *June 30, 2008*
Published: *July 21, 2008*

$\mathcal{N} = 4$ Superconformal Chern-Simons theories with hyper and twisted hyper multiplets

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Abstract: We extend the $\mathcal{N} = 4$ superconformal Chern-Simons theories of Gaiotto and Witten to those with additional twisted hyper-multiplets. The new theories are generically linear quiver gauge theories with the two types of hyper-multiplets alternating between gauge groups. Our construction includes the Bagger-Lambert model of SO(4) gauge group. A family of abelian theories are identified with those proposed earlier in the context of the M-crystal model for M2-branes probing $(\mathbb{C}^2/\mathbb{Z}_n)^2$ orbifolds. Possible extension with non-abelian BF couplings and string/M-theory realization are briefly discussed.

Keywords: Supersymmetric gauge theory, M-Theory, Brane Dynamics in Gauge Theories, Field Theories in Lower Dimensions.

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1. Introduction

Recently there has been considerable works on three-dimensional superconformal theories with larger supersymmetries. Bagger and Lambert (BL) has proposed an $\mathcal{N} = 8$ superconformal model with three-algebra structure and SO(8) R-symmetry as a theory on multiple M2 branes [1−5]. More recently Gaiotto and Witten (GW) has proposed $\mathcal{N} = 4$ superconformal Chern-Simons models with SO(4) R-symmetry classified by super-algebras [6].

In this work we generalize the Gaiotto-Witten's work to include twisted hyper-multiplets. Quiver theories appear naturally with two types of hyper-multiplets alternating between gauge groups where the quiver diagram is linear or circular with multiple nodes. The Bagger-Lambert theory with SO(4) gauge group appears naturally as a simplest kind of the quiver theory. Our work is partially motivated by attempt to understand the Bagger-Lambert theory with SO(4) gauge group in the context of the Gaiotto-Witten theory.

The number of supersymmetries of three-dimensional superconformal Chern-Simons theories has a natural division with $\mathcal{N} = 3$ [7]. It is rather straightforward to have the theories with $\mathcal{N} \leq 3$, and there has been some recent work on $\mathcal{N} = 2, 3$ superconformal theories [8]. For the conformal theory of M2 branes, one needs more supersymmetry [9] and the recent works related to the BL theory and the GW theory can be regarded as concrete steps toward this direction.

The BL theory was proposed as a superconformal field theory on coincident M2 branes. The BL theory is a superconformal Chern-Simons theory with maximal supersymmetry and SO(8) R-symmetry. It is parity even [10, 11] and its superconformal symmetry group is

OSp(8|4). The BL theory is supposed to describe two M2 branes sitting at the origin of $\mathbb{R}^8$ with certain discrete symmetry [12, 13]. For the additional many results on the BL theory, see [14 – 39].

Meanwhile, the GW theories are $\mathcal{N} = 4$ superconformal Chern-Simons theories with SO(4) R-symmetry and OSp(4|4) as the superconformal group. The GW theories have two gauge groups of opposite Chern-Simons coefficient, and the possible gauge fields are $\mathrm{U}(N) \times \mathrm{U}(M)$ or $O(N) \times \mathrm{Sp}(M)$. The matter field belongs to the bi-fundamental hyper-multiplet $(q^A_\alpha, \psi^A_{\dot\alpha})$ where the scalar field q^A_α and the spinor field $\psi^A_{\dot\alpha}$ belong to $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SO}(4)$ R-symmetry group.

The GW theories were developed as defect conformal field theories dual to the half-supersymmetric Janus geometry [40]. The four-dimensional Janus field theory can have at most eight supersymmetries, suggesting that the possible existence of the three-dimensional $\mathcal{N} = 4$ superconformal field theories. The GW theories form a complete class of the superconformal Chern-Simons theories with a single bi-fundamental hyper-multiplet.

On the other hand, the BL theory with SO(4) gauge group has SO(8) R-symmetry. If one keeps only a half of the supersymmetry, the R-symmetry group would be reduced to SO(4), and the matter field splits to one hyper-multiplet and one twisted hyper-multiplet. The twisted hyper-multiplet $(\tilde{q}^A_{\dot\alpha}, \tilde{\psi}^A_\alpha)$ belongs to $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{1})$ representations of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group. This suggests that there could be a generalization of the GW theories with additional twisted hyper-multiplet. Indeed, the present work realizes this possibility.

Another motivation for our work arises from consideration of the so-called M-crystal model [41 – 43] of $\mathrm{AdS}_4/\mathrm{CFT}_3$ dual pairs. This M-crystal model is the M2-brane counterpart of the famous brane-tiling model [45 – 47] for D3-brane gauge theories. The M-crystal model graphically encodes certain information on the world-volume theories of M2-branes probing toric Calabi-Yau four-fold (CY_4) cones. When the CY_4 cone is a product of two singular ALE spaces, the world-volume theory has $\mathcal{N} = 4$ supersymmetry.

In ref. [43], it was shown how to derive an abelian gauge theory from a given M-crystal model. In particular, for $\mathrm{CY}_4 = (\mathbb{C}^2/\mathbb{Z}_n)^2$, it was found that the abelian theory must contain both types of hyper-multiplets, for a somewhat similar reason to the BL theory case. The derivation of [43] however was incomplete, as the kinetic terms for the abelian gauge fields were not determined. Nevertheless, based on the analysis of the moduli space of vacua, the gauge fields were argued to be non-dynamical. Naturally, our work provides a possible candidate theory for the corresponding CFT_3.

In section 2, we begin by reviewing the GW construction. In this construction, one uses the $\mathcal{N} = 1$ theory with an SO(3) global symmetry and adjusts the coupling constants to enhance the global symmetry to the SO(4) R-symmetry, which does not commute with the $\mathcal{N} = 1$ supercharge. The resulting theory has $\mathcal{N} = 4$ supersymmetry. We employ a similar method to include additional twisted hyper-multiplet to the theory and find new $\mathcal{N} = 4$ superconformal field theories.

As in ref. [6], the gauge groups and the matter contents are classified by certain super-algebras. The restriction is quite severe, so that to have non-trivial interactions between the two types of hyper-multiplets, the theory must be a linear quiver gauge theory (possibly forming a closed loop) with the two types of hyper-multiplets alternating between neighboring gauge groups. We also present how to take a mass deformation of the general $\mathcal{N}=4$ theories without breaking any supersymmetry or SO(4) R-symmetry.

In section 3, we discuss how our construction is related to other works. First, we show explicitly that the BL model of SO(4) gauge group with its mass deformation [21, 23] is a special case of our construction. For this case, the supersymmetry is doubled by chance. However, we suspect that our construction for two types of hyper-multiplets gets an enhanced supersymmetry only for the BL case. The second half is devoted to the connection to the M-crystal model. After a short review of the M-crystal model specialized to $\mathcal{N}=4$, we show that a particular abelian quiver theory of the present paper can be identified with the one proposed in [43] for M2-branes probing $(\mathbb{C}^2/\mathbb{Z}_n)^2$ orbifolds.

We conclude with some discussions on future directions in section 4.

2. $\mathcal{N}=4$ Chern-Simons theories with two types of hyper-multiplets

In this section, we start with a brief review of the GW construction of the $\mathcal{N}=4$ superconformal theories with only hyper-multiplets, and then present a generalization to include twisted hyper-multiplets.

2.1 Gaiotto-Witten revisited

We start with an Sp($2n$) group and let A, B indices run over a $2n$-dimensional representation. We denote the anti-symmetric invariant tensor of Sp($2n$) by ω_{AB} and choose all the generators $t^A{}_B$ to be anti-Hermitian $(2n\times 2n)$ matrices, such that $(\omega_{AC}t^C{}_B)$ are symmetric matrices. We consider a Chern-Simons gauge theory whose gauge group is a subgroup of Sp($2n$) and we denote anti-Hermitian generators of the gauge group as $(t^m)^A{}_B$ which satisfy

$$[t^m, t^n] = f^{mn}{}_p t^p.$$

Gauge field and gaugino are denoted by $(A_m)_\mu$ and χ_m, and the adjoint indices are raised or lowered by an invariant quadratic form k^{mn} or its inverse k_{mn} of the gauge group. We will also use

$$\chi^A{}_B = \chi_m (t^m)^A{}_B, \qquad \chi_{AB} = \chi_m \omega_{AC}(t^m)^C{}_B = \chi_m t^m_{AB}.$$

We couple the gauge theory with a hyper-multiplet matter fields $(q^A_\alpha, \psi^A_{\dot\alpha})$ satisfying the reality condition

$$\bar q^\alpha_A = (q^A_\alpha)^\dagger = \epsilon^{\alpha\beta}\omega_{AB}q^B_\beta, \qquad \bar\psi^A_{\dot\alpha} = (\psi^A_{\dot\alpha})^\dagger = \epsilon^{\dot\alpha\dot\beta}\omega_{AB}\psi^B_{\dot\beta}.$$

We use $(\alpha,\beta;\dot\alpha,\dot\beta)$ doublet indices for the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry group.

To get $\mathcal{N}=4$ supersymmetric theories, it is convenient to work in the $\mathcal{N}=1$ framework in which $(q^A_\alpha, \psi^A_{\dot\alpha}, F^A_\alpha)$ and $((A_m)_\mu, \chi_m)$ are the basic supermultiplets. Our conventions for spinors and $\mathcal{N}=1$ superfields are summarized in appendix A. We start from the general $\mathcal{N}=1$ Lagrangian with a global SU(2) symmetry which acts on the indices (α,β) and

$(\dot\alpha,\dot\beta)$ simultaneously. Such a Lagrangian would take the form $\mathcal{L}=\mathcal{L}_{\rm CS}+\mathcal{L}_{\rm kin}+\mathcal{L}_W$, where

$$\begin{aligned}
\mathcal{L}_{\rm CS} &= \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}k_{mn}A^m_\mu\partial_\nu A^n_\lambda+\frac{\varepsilon^{\mu\nu\lambda}}{12\pi}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda+\frac{ik_{mn}}{4\pi}\chi^m\chi^n, \qquad (2.1)\\
\mathcal{L}_{\rm kin} &= \frac{1}{2}\left(-D\bar q^\alpha_A Dq^A_\alpha+\bar F^\alpha_A F^A_\alpha+i\bar\psi^{\dot\alpha}_A \not{D}\psi^A_{\dot\alpha}-i\bar\psi^{\dot\alpha}_A\chi^A{}_B q^B_\alpha+i\bar q^\alpha_A\chi^A{}_B\psi^B_{\dot\alpha}\right)\\
&= \frac{1}{2}\omega_{AB}\left[\epsilon^{\alpha\beta}\left(-Dq^A_\alpha Dq^B_\beta+F^A_\alpha F^B_\beta\right)+i\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}\not{D}\psi^B_{\dot\beta}\right]-i\epsilon^{\dot\alpha\beta}\psi^A_{\dot\alpha}\chi_{AB}q^B_\beta,\\
\mathcal{L}_W &= -\pi T_{AB,CD}\left(\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}F^A_\alpha q^B_\beta q^C_\gamma q^D_\delta+\frac{i}{2}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\psi^A_{\dot\alpha}\psi^B_{\dot\beta}q^C_\gamma q^D_\delta+i\epsilon^{\dot\alpha\beta}\epsilon^{\dot\gamma\delta}\psi^A_{\dot\alpha}q^B_\beta\psi^C_{\dot\gamma}q^D_\delta\right),
\end{aligned}$$

where $D_\mu q^A_\alpha=\partial_\mu q^A_\alpha+A_{m\mu}(t^m)^A{}_B q^B_\alpha$. The superpotential takes the form

$$W=\frac{\pi}{4}T_{AB,CD}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}q^A_\alpha q^B_\beta q^C_\gamma q^D_\delta. \qquad (2.2)$$

Here the superpotential coupling $T_{AB,CD}$ is anti-symmetric under the permutation of A,B or C,D, and symmetric under the exchange of the pairs (AB) with (CD). For a suitable choice of $T_{AB,CD}$, the Lagrangian becomes invariant under two SU(2) rotations that act on q^A_α and $\psi^A_{\dot\alpha}$ separately and therefore $\mathcal{N}=4$ supersymmetric. Let us now examine each part of the Lagrangian which needs to be separately invariant under the SO(4).

Yukawa coupling. As the gaugino field χ_m is a purely auxiliary field, we integrate it out to get a $(q^2\psi^2)$ term,

$$\begin{aligned}
\mathcal{L} &= \frac{ik_{mn}}{4\pi}\chi^m\chi^n-i\chi_m\left(\epsilon^{\dot\alpha\beta}\psi^A_{\dot\alpha}t^m_{AB}q^B_\beta\right)+\cdots\\
&= \frac{ik_{mn}}{4\pi}\left(\chi^m-2\pi\epsilon^{\dot\alpha\beta}\psi^A_{\dot\alpha}t^m_{AB}q^B_\beta\right)\left(\chi^n-2\pi\epsilon^{\dot\gamma\delta}\psi^C_{\dot\gamma}t^n_{CD}q^D_\delta\right)\\
&\quad -i\pi k_{mn}t^m_{AB}t^n_{CD}\epsilon^{\dot\alpha\beta}\epsilon^{\dot\gamma\delta}\psi^A_{\dot\alpha}q^B_\beta\psi^C_{\dot\gamma}q^D_\delta+\cdots. \qquad (2.3)
\end{aligned}$$

Combining this with the other $(q^2\psi^2)$ terms arising from the superpotential $\mathcal{L}_W$, one finds

$$-i\pi q^A_\alpha q^B_\beta\psi^C_{\dot\gamma}\psi^D_{\dot\delta}\left(k_{mn}t^m_{AC}t^n_{BD}\epsilon^{\alpha\dot\gamma}\epsilon^{\beta\dot\delta}+\tfrac{1}{2}T_{AB,CD}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}+T_{AC,BD}\epsilon^{\alpha\dot\gamma}\epsilon^{\beta\dot\delta}\right). \qquad (2.4)$$

This expression has to be $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$-invariant on its own. It implies that the terms containing contractions between dotted and undotted indices must vanish. For the Yukawa coupling, it suffices to require that the part proportional to $q^{(A}_{(\alpha}q^{B)}_{\beta)}$ should vanish:

$$T_{AC,BD}+T_{BC,AD}+k_{mn}\left(t^m_{AC}t^n_{BD}+t^m_{BC}t^n_{AD}\right) \;=\; 0. \qquad (2.5)$$

In ref. [6], this equation was shown to determine the coupling constants,

$$T_{AB,CD} \;=\; \frac{1}{3}k_{mn}\left(t^m_{AC}t^n_{BD}-t^m_{BC}t^n_{AD}\right). \qquad (2.6)$$

and imposes a constraint on t^m_{AB} which the authors called the “fundamental identity”,

$$k_{mn}t^m_{(AB}t^n_{C)D}=0, \qquad (2.7)$$

where the indices A, B, C are symmetrized over cyclic permutations. In ref. [6], it was also noticed that this identity can be understood as the Jacobi identity for three fermionic generators of a super Lie algebra,

$$[M^m, M^n] = f^{mn}{}_p M^p, \qquad [M^m, Q_A] = Q_B (t^m)^B{}_A, \qquad \{Q_A, Q_B\} = t^m_{AB} M_m. \tag{2.8}$$

This turns out to be a rather strong constraint on the field content of the theory. Namely, the gauge group and matter should be such that the gauge symmetry algebra can be extended to a super Lie algebra by adding fermionic generators associated to hyper-multiplets. The final expression for the $(q^2\psi^2)$ term in the Lagrangian is

$$\begin{aligned}\mathcal{L}_{q^2\psi^2} &= -i\pi q^A_\alpha q^B_\beta \psi^C_\gamma \psi^D_\delta \epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta} k_{mn} t^m_{AC} t^n_{BD} \\ &= -i\pi k_{mn} \epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta} \jmath^m_{\alpha\dot\gamma} \jmath^n_{\beta\dot\delta} \qquad \left(\jmath^m_{\alpha\dot\gamma} \equiv q^A_\alpha t^m_{AC} \psi^C_{\dot\gamma}\right).\end{aligned} \tag{2.9}$$

Bosonic potential. We present here the computation of bosonic potential in some detail for later convenience. In terms of the "moment map", $\mu^m_{\alpha\beta} \equiv t^m_{AB} q^A_\alpha q^B_\beta$ [6], we can write

$$W = \frac{\pi}{6}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} k_{mn} \mu^m_{\alpha\gamma}\mu^n_{\beta\delta}. \tag{2.10}$$

To compute the bosonic potential, it is useful to note

$$\omega^{AB}\epsilon_{\alpha\beta}\frac{\partial \mu^m_{\gamma\delta}}{\partial q^A_\alpha}\frac{\partial \mu^n_{\kappa\rho}}{\partial q^B_\beta} = \epsilon_{\gamma\kappa}\mu^{mn}_{\delta\rho} + \epsilon_{\gamma\rho}\mu^{mn}_{\delta\kappa} + \epsilon_{\delta\kappa}\mu^{mn}_{\gamma\rho} + \epsilon_{\delta\rho}\mu^{mn}_{\gamma\kappa}, \tag{2.11}$$

where

$$\mu^{mn}_{\alpha\beta} \equiv (\omega t^m t^n)_{AB} q^A_\alpha q^B_\beta.$$

The potential term is

$$\begin{aligned} V &= \frac{1}{2}\epsilon_{\alpha\beta}\omega^{AB}\frac{\partial W}{\partial q^A_\alpha}\frac{\partial W}{\partial q^B_\beta} = \frac{2\pi^2}{9}\epsilon_{\alpha\gamma}\mu^{mn}_{\beta\delta}\mu^{\alpha\beta}_m \mu^{\gamma\delta}_n \\ &= \frac{\pi^2}{9}\epsilon_{\alpha\gamma} f^{mnp}\mu_{p,\beta\delta}\mu^{\alpha\beta}_m\mu^{\gamma\delta}_n + \frac{\pi^2}{9}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\epsilon_{\kappa\rho}\mu^{mn,\kappa\rho}\mu^{\alpha\beta}_m\mu^{\gamma\delta}_n. \end{aligned} \tag{2.12}$$

As explained in ref. [6], we apply the fundamental identity (2.7) to rotate the upper indices $(\alpha\beta\kappa)$ in the second term. After some manipulations, we find

$$V = \frac{\pi^2}{9}\epsilon_{\alpha\gamma} f^{mnp}\mu_{p,\beta\delta}\mu^{\alpha\beta}_m\mu^{\gamma\delta}_n - \frac{2\pi^2}{9}\epsilon_{\alpha\gamma}\mu^{nm}_{\beta\delta}\mu^{\alpha\beta}_m\mu^{\gamma\delta}_n. \tag{2.13}$$

Taking the average of the first line of (2.12) and (2.13), we arrive at the final expression,

$$V = \frac{\pi^2}{6} f_{mnp} (\mu^m)^\alpha{}_\beta (\mu^n)^\beta{}_\gamma (\mu^p)^\gamma{}_\alpha. \tag{2.14}$$

Full theory. To summarize, the Gaiotto-Witten theory in a manifestly $\mathcal{N}=4$ supersymmetric notation without auxiliary fields consists of the Lagrangian

$$\mathcal{L} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right) + \frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta}Dq^A_\alpha Dq^B_\beta + i\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}\not{D}\psi^B_{\dot\beta}\right)$$
$$-i\pi k_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}j^m_{\alpha\dot\gamma}j^n_{\beta\dot\delta} - \frac{\pi^2}{6}f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\mu^p)^\gamma{}_\alpha, \tag{2.15}$$

and the supersymmetry transformation rules

$$\delta q^A_\alpha = i\eta_\alpha{}^{\dot\alpha}\psi^A_{\dot\alpha}, \qquad \delta A^m_\mu = 2\pi i\eta^{\alpha\dot\alpha}\gamma_\mu j^m_{\alpha\dot\alpha},$$
$$\delta\psi^A_{\dot\alpha} = \left[\not{D}q^A_\alpha + \frac{2\pi}{3}(t_m)^A{}_B q^B_\beta(\mu^m)^\beta{}_\alpha\right]\eta^\alpha{}_{\dot\alpha}. \tag{2.16}$$

The supersymmetry parameter η is a Majorana spinor and transform in the $(\mathbf{2},\mathbf{2})$ representation of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$.

The classical supergroups related to the GW models are $\mathrm{U}(N|M)$ and $\mathrm{OSp}(N|M)$. The gauge groups are product groups $\mathrm{U}(N)\times\mathrm{U}(M)$ or $O(N)\times\mathrm{Sp}(M)$ with equal and opposite Chern-Simons coefficients for the two factors. The matter fields belong to a bi-fundamental representation of the product gauge group. Of course, one can have a multiple embedding of t^m in $\mathrm{Sp}(2n)$, resulting in many copies of the GW models, possibly with different gauge group pairs, and no coupling between different blocks of GW models.

2.2 Adding twisted hyper-multiplets

Let us try to include the twisted hyper-multiplets. We denote them by $(\tilde{q}^A_{\dot\alpha}, \tilde\psi^A_\alpha, \tilde{F}^A_{\dot\alpha})$ and define their moment map and its super-partner [6] as in the untwisted case.

$$\tilde\mu^m_{\dot\alpha\dot\beta} \equiv \tilde{t}^m_{AB}\tilde{q}^A_{\dot\alpha}\tilde{q}^B_{\dot\beta}, \qquad \tilde{j}^m_{\dot\alpha\alpha} \equiv \tilde{q}^A_{\dot\alpha}\tilde{t}^m_{AB}\tilde\psi^B_\alpha. \tag{2.17}$$

Both types of hyper-multiplets share the same gauge symmetry, so the structure constants $f^{mn}{}_p$ and the quadratic form k^{mn} are identical. However, they can take different representations, so in general the generators $\tilde{t}^m$ are different from t^m. Strictly speaking, we should distinguish the A,B indices of hyper-multiplets from those of twisted hyper-multiplets, but we suppress the distinction to avoid clutter.

The construction of $\mathcal{L}_{\rm CS}$ and $\mathcal{L}_{\rm kin}$ proceeds in the same way as before. The Yukawa and bosonic potential terms become more complicated due to the mixing of the hyper and twisted hyper-multiplets. We need to make those terms $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetric by a suitable choice of the superpotential. The unmixed terms can all be made R-symmetric by introducing the super-potential

$$W_0 = \frac{\pi}{6}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}k_{mn}\mu^m_{\alpha\gamma}\mu^n_{\beta\delta} + \frac{\pi}{6}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}k_{mn}\tilde\mu^m_{\dot\alpha\dot\gamma}\tilde\mu^n_{\dot\beta\dot\delta}, \tag{2.18}$$

provided that the fundamental identity (2.7) holds for both of the matrices t^m_{AB} and $\tilde{t}^m_{AB}$. The hyper and twisted hyper-multiplets therefore give two (in general independent) extensions of the gauge symmetry algebra to a super Lie algebra.

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There may be many copies of the simple GW models in many pairs of the gauge groups related to the supergroups. The key point is that the gauge group pairs for the twisted hyper-multiplets do not need to coincide with those for the hyper-multiplets. This allows the interaction between different copies of the GW model for hyper-multiplets, leading to a quiver theory. While there may be many sets of quivers, one can focus on one irreducible quiver theory where every part of theory are interacting with each other.

Now we focus on possible R-symmetry breaking terms due to the mixed interactions.

Yukawa coupling. We first integrate out the gaugino,

$$\chi^m = 2\pi\left(\epsilon^{\dot\alpha\beta}\psi^A_{\dot\alpha} t^m_{AB} q^B_\beta + \epsilon^{\alpha\dot\beta}\tilde\psi^A_\alpha \tilde t^m_{AB}\tilde q^B_{\dot\beta}\right). \tag{2.19}$$

Computing the Yukawa term, we find a cross term from the gaugino squared,

$$\mathcal{L}_{\text{Yukawa}} \;=\; -2\pi i\epsilon^{\alpha\dot\beta}\epsilon^{\dot\gamma\delta}k_{mn}t^m_{AB}\tilde t^n_{CD} q^A_\alpha \psi^B_{\dot\beta}\tilde q^C_{\dot\gamma}\tilde\psi^D_\delta + (\text{unmixed}), \tag{2.20}$$

which is not $\mathrm{SO}(4)_R$ invariant by itself. To restore the R-symmetry, we try adding some mixed terms to the superpotential. The most general form allowed by the diagonal SU(2) and gauge symmetries is

$$\Delta W \;=\; \pi\tilde S_{AB,CD}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}q^A_\alpha q^B_\beta \tilde q^C_{\dot\gamma}\tilde q^D_{\dot\delta} + \pi S_{AB,CD}\epsilon^{\alpha\dot\gamma}\epsilon^{\beta\dot\delta} q^A_\alpha q^B_\beta\tilde q^C_{\dot\gamma}\tilde q^D_{\dot\delta}, \tag{2.21}$$

where the coupling constants $\tilde S$ and S satisfy

$$\tilde S_{AB,CD} = -\tilde S_{BA,CD} = -\tilde S_{AB,DC}, \qquad S_{AB,CD} = S_{BA,CD} = S_{AB,DC}.$$

The additional superpotential yields some Yukawa terms which are themselves R-invariant,

$$\begin{aligned}\Delta\mathcal{L}^{(1)}_{\text{Yukawa}} = &-i\pi\tilde S_{AB,CD}\left(\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}\psi^A_{\dot\alpha}\psi^B_{\dot\beta}\tilde q^C_{\dot\gamma}\tilde q^D_{\dot\delta} + \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}q^A_\alpha q^B_\beta\tilde\psi^C_\gamma\tilde\psi^D_\delta\right) \\ &-i\pi S_{AB,CD}\left(\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}\psi^A_{\dot\alpha}\psi^B_{\dot\beta}\tilde q^C_{\dot\gamma}\tilde q^D_{\dot\delta} + \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}q^A_\alpha q^B_\beta\tilde\psi^C_\gamma\tilde\psi^D_\delta + 2\epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\delta}q^A_\alpha\psi^B_{\dot\beta}\tilde\psi^C_\gamma\tilde q^D_{\dot\delta}\right),\end{aligned} \tag{2.22}$$

and those which are not,

$$\Delta\mathcal{L}^{(2)}_{\text{Yukawa}} \;=\; -2\pi i\left(2\tilde S_{AB,CD}\epsilon^{\alpha\dot\beta}\epsilon^{\dot\gamma\delta} + S_{AB,CD}\epsilon^{\alpha\dot\gamma}\epsilon^{\dot\beta\delta}\right) q^A_\alpha\psi^B_{\dot\beta}\tilde q^C_{\dot\gamma}\tilde\psi^D_\delta. \tag{2.23}$$

It is possible to combine (2.20) and (2.23) in an R-symmetric way,

$$k_{mn}t^m_{AB}\tilde t^n_{CD}\epsilon^{\alpha\dot\beta}\epsilon^{\dot\gamma\delta} + 2\tilde S_{AB,CD}\epsilon^{\alpha\dot\beta}\epsilon^{\dot\gamma\delta} + S_{AB,CD}\epsilon^{\alpha\dot\gamma}\epsilon^{\dot\beta\delta} \;\sim\; \epsilon^{\alpha\delta}\epsilon^{\dot\beta\dot\gamma}, \tag{2.24}$$

by using an identity for the diagonal SU(2): $\epsilon^{\alpha\dot\beta}\epsilon^{\dot\gamma\delta} + \epsilon^{\alpha\dot\gamma}\epsilon^{\delta\dot\beta} + \epsilon^{\alpha\delta}\epsilon^{\dot\beta\dot\gamma} = 0$. That uniquely determines the two coupling constants of ΔW:

$$\tilde S_{AB,CD} = 0, \qquad S_{AB,CD} = -k_{mn}t^m_{AB}\tilde t^n_{CD}. \tag{2.25}$$

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Bosonic potential. The superpotential is $W = W_0 + \Delta W$, where

$$W_0 = \frac{\pi}{6}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}k_{mn}\mu^m_{\alpha\gamma}\mu^n_{\beta\delta} + \frac{\pi}{6}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}k_{mn}\tilde\mu^m_{\dot\alpha\dot\gamma}\tilde\mu^n_{\dot\beta\dot\delta}, \tag{2.26}$$

$$\Delta W = -\pi\epsilon^{\alpha\dot\beta}\epsilon^{\gamma\dot\delta}k_{mn}\mu^m_{\alpha\gamma}\tilde\mu^n_{\dot\beta\dot\delta}. \tag{2.27}$$

We need to compute the mixed term $V|_{q^4\tilde q^2}$ in the bosonic potential and check its R-invariance. The computation of the other mixed term $V|_{q^2\tilde q^4}$ is similar. There are two contributions to $V|_{q^4\tilde q^2}$,

$$\begin{aligned} V|_{q^4\tilde q^2} &= \epsilon_{\alpha\beta}\omega^{AB}\frac{\partial W_0}{\partial q^A_\alpha}\frac{\partial \Delta W}{\partial q^B_\beta} + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\omega^{AB}\frac{\partial \Delta W}{\partial \tilde q^A_{\dot\alpha}}\frac{\partial \Delta W}{\partial \tilde q^B_{\dot\beta}} \\ &= -\frac{4\pi^2}{3}\epsilon_{\alpha\dot\gamma}\mu^{mn}_{\beta\delta}\mu^{\alpha\beta}_m\tilde\mu^{\dot\gamma\delta}_n + 2\pi^2\epsilon_{\alpha\gamma}\tilde\mu^{mn}_{\dot\beta\dot\delta}\mu^{\alpha\beta}_m\mu^{\gamma\delta}_n. \end{aligned} \tag{2.28}$$

Using the trick explained below (2.12) again, we can rewrite the first term as

$$-\pi^2 f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\tilde\mu^p)^{\dot\gamma}{}_{\dot\alpha}.$$

The second term is decomposed into those proportional to $\tilde\mu^{[mn]}_{(\dot\beta\dot\delta)}$ or $\tilde\mu^{(mn)}_{[\dot\beta\dot\delta]}$. The former precisely cancels against the above (first) term, and we are left with the latter which is indeed R-invariant,

$$V|_{q^4\tilde q^2} = -\pi^2(\tilde\mu^{mn})^{\dot\gamma}{}_{\dot\gamma}(\mu_m)^\alpha{}_\beta(\mu_n)^\beta{}_\alpha. \tag{2.29}$$

Combining all the mixed and unmixed terms, we obtain the full bosonic potential,

$$\begin{aligned} V = &\frac{\pi^2}{6}f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\mu^p)^\gamma{}_\alpha + \frac{\pi^2}{6}f_{mnp}(\tilde\mu^m)^\alpha{}_\beta(\tilde\mu^n)^\beta{}_\gamma(\tilde\mu^p)^\gamma{}_\alpha \\ &-\pi^2(\tilde\mu^{mn})^{\dot\gamma}{}_{\dot\gamma}(\mu_m)^\alpha{}_\beta(\mu_n)^\beta{}_\alpha - \pi^2(\mu^{mn})^\gamma{}_\gamma(\tilde\mu_m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu_n)^{\dot\beta}{}_{\dot\alpha}. \end{aligned} \tag{2.30}$$

Full theory. In summary, we have found the generalization of Gaiotto-Witten theory which includes both hyper and twisted hyper-multiplets. The full Lagrangian is given by

$$\begin{aligned} \mathcal{L} = &\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right) \\ &+\frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta}Dq^A_\alpha Dq^B_\beta + i\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}\not{D}\psi^B_{\dot\beta}\right) + \frac{1}{2}\tilde\omega_{AB}\left(-\epsilon^{\dot\alpha\dot\beta}D\tilde q^A_{\dot\alpha}D\tilde q^B_{\dot\beta} + i\epsilon^{\alpha\beta}\tilde\psi^A_\alpha\not{D}\tilde\psi^B_\beta\right) \\ &-i\pi k_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}j^m_{\alpha\dot\gamma}j^n_{\beta\dot\delta} - i\pi k_{mn}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\tilde j^m_{\dot\alpha\gamma}\tilde j^n_{\dot\beta\delta} + 4\pi i k_{mn}\epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\delta}j^m_{\alpha\dot\beta}\tilde j^n_{\dot\delta\gamma} \\ &+i\pi k_{mn}\left(\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}\tilde\mu^m_{\dot\alpha\dot\beta}\psi^A_{\dot\gamma}t^n_{AB}\psi^B_{\dot\delta} + \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mu^m_{\alpha\beta}\tilde\psi^A_\gamma\tilde t^n_{AB}\tilde\psi^B_\delta\right) \\ &-\frac{\pi^2}{6}f_{mnp}(\tilde\mu^m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu^n)^{\dot\beta}{}_{\dot\gamma}(\tilde\mu^p)^{\dot\gamma}{}_{\dot\alpha} - \frac{\pi^2}{6}f_{mnp}(\tilde\mu^m)^\alpha{}_\beta(\tilde\mu^n)^\beta{}_\gamma(\tilde\mu^p)^\gamma{}_\alpha \\ &+\pi^2(\tilde\mu^{mn})^{\dot\gamma}{}_{\dot\gamma}(\mu_m)^\alpha{}_\beta(\mu_n)^\beta{}_\alpha + \pi^2(\mu^{mn})^\gamma{}_\gamma(\tilde\mu_m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu_n)^{\dot\beta}{}_{\dot\alpha}, \end{aligned} \tag{2.31}$$

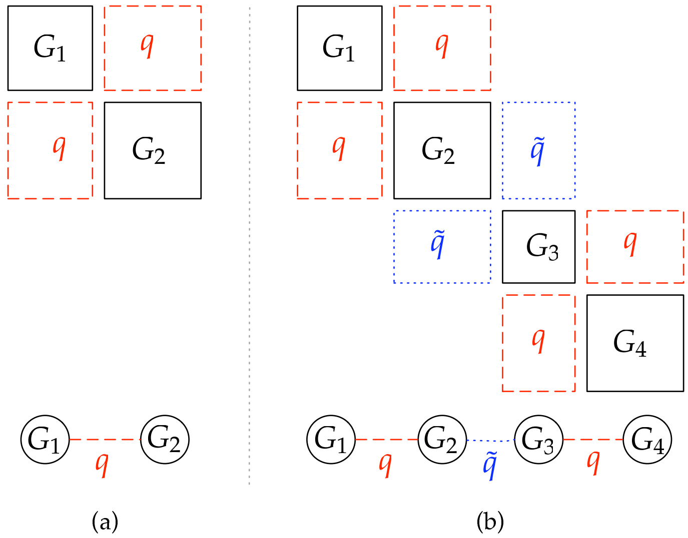

Figure 1: Linear quiver structure of the original and extended GW theories. The gauge groups can be either $\mathrm{U}(N_i)$ or alternation between $O(N_i)$ and $\mathrm{Sp}(M_i)$. Dashed lines denote hyper-multiplets and dotted lines denote twisted hyper-multiplets.

and the supersymmetry transformation law is

$$\begin{aligned}
\delta q^A_\alpha &= +i\eta_\alpha{}^{\dot\alpha}\psi^A_{\dot\alpha}, \\
\delta \tilde{q}^A_{\dot\alpha} &= -i\eta^\alpha{}_{\dot\alpha}\tilde{\psi}^A_\alpha, \\
\delta A^m_\mu &= 2\pi i\eta^{\alpha\dot\alpha}\gamma_\mu(j^m_{\alpha\dot\alpha} - \tilde{j}^m_{\dot\alpha\alpha}), \\
\delta\psi^A_{\dot\alpha} &= +\left[\not{D} q^A_\alpha + \frac{2\pi}{3}(t_m)^A{}_B q^B_\beta(\mu^m)^\beta{}_\alpha\right]\eta^\alpha{}_{\dot\alpha} - 2\pi(t_m)^A{}_B q^B_\beta(\tilde{\mu}^m)^{\dot\beta}{}_{\dot\alpha}\eta^\beta{}_{\dot\beta}, \\
\delta\tilde{\psi}^A_\alpha &= -\left[\not{D}\tilde{q}^A_{\dot\alpha} + \frac{2\pi}{3}(\tilde{t}_m)^A{}_B\tilde{q}^B_{\dot\beta}(\tilde{\mu}^m)^{\dot\beta}{}_{\dot\alpha}\right]\eta_\alpha{}^{\dot\alpha} + 2\pi(\tilde{t}_m)^A{}_B\tilde{q}^B_{\dot\beta}(\mu^m)^\beta{}_\alpha\eta_\beta{}^{\dot\beta}.
\end{aligned} \tag{2.32}$$

Classification in terms of quivers. When only hyper-multiplets are considered, the classification of $\mathcal{N} = 4$ Chern-Simons-matter theories boils down to that of super Lie algebras. In the purely non-abelian case, the only possibilities are either the basic models $\mathrm{U}(N|M)$ and $\mathrm{OSp}(N|M)$ or multiple copies of them; see figure 1(a). Naive attempts to obtain more general quiver theories by connecting several gauge groups with bi-fundamental hyper-multiplets immediately ruin the super Lie algebra structure.

Once both hyper and twisted hyper-multiplets are included, we have a richer variety of theories characterized by quiver diagrams depicted in figure 1(b). If we align the gauge group factors linearly and introduce bi-fundamental matter fields such that the two types of hyper-multiplets alternate among the gauge groups, we can have an interacting theory of both types of hyper-multiplets without violating the fundamental identity (2.7). Note that the hyper and twisted hyper-multiplets give two different ways to pair the gauge groups into supergroups. We should emphasize that, apart from trivial direct sums, these linear quiver theories exhaust all possible (purely non-abelian) theories.

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The linear quiver can either have open ends or form a closed circular loop. With $\mathrm{U}(N_i)$ gauge groups, the relevant quiver diagrams are the Dynkin diagrams for A_n or $\hat{A}_{2n-1}$, respectively. The BL-like models, which have only two gauge group factors and both hyper-multiplets in the same representation, can be thought of as the shortest closed loops.

Abelian CS/BF theories. Abelian theories deserve separate treatment. The theories are defined by $\mathrm{U}(1)$ gauge fields A_m, quaternion-valued (twisted) hyper-multiplets $(q^i, \tilde{q}^{\bar{\imath}})$, the charge matrices $(Q_i^m, \widetilde{Q}_{\bar{\imath}}^m)$, and the CS coefficient k^{mn}. The fundamental identity (2.7) now reads

$$k_{mn} Q_i^m Q_j^n = 0, \qquad k_{mn} \widetilde{Q}_{\bar{\imath}}^m \widetilde{Q}_{\bar{\jmath}}^n = 0. \tag{2.33}$$

The constraint is less restrictive than in the non-abelian case, so we have more freedom to construct new theories. We will not try to obtain a catalogue of all possible such theories. Instead, we will consider two typical solutions to the constraint (2.33).

The first solution is to employ the linear quiver structure discussed in the non-abelian case. The constraint (2.33) is trivially satisfied because only one hyper-multiplet and one twisted hyper-multiplet (and not two hyper-multiplets) meet at the same gauge group. Unlike the non-abelian case, however, the values of the charges can differ from ± 1.

There is another simple way to satisfy the constraint (2.33). Divide the gauge fields into two groups, say A_m and $\tilde{A}_{\bar{m}}$, and demand that q^i be charged only under A_m and $\tilde{q}^{\bar{\imath}}$ only under $\tilde{A}_{\bar{m}}$. If we further demand that $k_{mn} = 0 = k_{\bar{m}\bar{n}}$ with $k_{\bar{m}n} \neq 0$, then the constraint (2.33) is trivially satisfied. The Chern-Simons term in the Lagrangian now reads

$$\mathcal{L}_{\mathrm{CS}} \sim k^{\bar{m}n} \tilde{A}_{\bar{m}} dA_n. \tag{2.34}$$

This type of coupling is more commonly known as abelian BF coupling ($\tilde{A} \to B$, $dA \to F$). In fact, it has been known for a long time [49] that abelian BF theories interacting with mater fields can be rendered $\mathcal{N} = 4$ supersymmetric. This is an exception (the only one we are aware of) to the $\mathcal{N} = 3$ threshold mentioned in the Introduction which predates the recent discoveries of non-abelian $\mathcal{N} \geq 4$ models.

The successes of the BL model and the GW construction partly rely on giving up the notion of a vector multiplet. In the absence of Yang-Mills kinetic term, the gauge field does not give rise to on-shell physical states, so it is permissible to have only the gauge field and none of its super-partners in the Lagrangian. In contrast, the abelian BF model of [49] maintains the whole $\mathcal{N} = 4$ vector-multiplet structure and is compatible with the Yang-Mills kinetic term.

We use the following notation for the (twisted) vector multiplets:

$$\begin{aligned} \text{vector}: &\quad (A_m, \chi_m^{\dot{\alpha}\beta}, s_m^{\dot{\alpha}\dot{\beta}} \; ; \; D_m^{\alpha\beta}), \\ \text{twisted vector}: &\quad (\tilde{A}_{\bar{m}}, \tilde{\chi}_{\bar{m}}^{\alpha\dot{\beta}}, \tilde{s}_{\bar{m}}^{\alpha\beta} \; ; \; \tilde{D}_{\bar{m}}^{\dot{\alpha}\dot{\beta}}), \end{aligned}$$

where $\chi(\tilde{\chi})$ is the gaugino, $s(\tilde{s})$ is the scalar triplet and $D(\tilde{D})$ is the triplet of auxiliary fields. The gauge index is raised and lowered by $k_{\bar{m}n}$; for example, $\tilde{\chi}^n \equiv k^{n\bar{m}} \tilde{\chi}_{\bar{m}}$. In the

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notation of the present paper, the Lagrangian of the abelian BF theory is given by [1]

$$
\begin{aligned}
\mathcal{L} = &\frac{\varepsilon^{\mu\nu\lambda}}{4\pi} k_{\bar{m}n} \tilde{A}^{\bar{m}}_{\mu} \partial_\nu A^n_\lambda + \frac{1}{4\pi}\left(i\tilde{\chi}^n_{\alpha\dot{\beta}} \chi^{\dot{\beta}\alpha}_n + \tilde{s}^n_{\alpha\beta} D^{\alpha\beta}_n + \tilde{D}^n_{\dot{\alpha}\dot{\beta}} s^{\dot{\alpha}\dot{\beta}}_n \right) \\
&+\frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta} D q^A_\alpha D q^B_\beta + i\epsilon^{\dot{\alpha}\dot{\beta}} \psi^A_{\dot{\alpha}} \not{D} \psi^B_{\dot{\beta}} \right) + \frac{1}{2}\tilde{\omega}_{AB}\left(-\epsilon^{\dot{\alpha}\dot{\beta}} D\tilde{q}^A_{\dot{\alpha}} D\tilde{q}^B_{\dot{\beta}} + i\epsilon^{\alpha\beta} \tilde{\psi}^A_\alpha \not{D} \tilde{\psi}^B_\beta \right) \\
&-\frac{i}{2} s^{\dot{\alpha}\dot{\beta}}_n (\psi\psi)^n_{\dot{\alpha}\dot{\beta}} - \frac{i}{2}\tilde{s}^{\alpha\beta}_{\bar{n}} (\tilde{\psi}\tilde{\psi})^{\bar{n}}_{\alpha\beta} - \frac{1}{4}(\mu^{mn})^\alpha{}_\alpha s^{\dot{\alpha}\dot{\beta}}_m s_{m\dot{\alpha}\dot{\beta}} - \frac{1}{4}(\tilde{\mu}^{mn})^{\dot{\alpha}}{}_{\dot{\alpha}} \tilde{s}^{\alpha\beta}_m \tilde{s}_{m\alpha\beta} \\
&+\chi^{\dot{\alpha}\beta}_n \jmath^n_{\beta\dot{\alpha}} + \tilde{\chi}^{\alpha\dot{\beta}}_n \tilde{\jmath}^n_{\dot{\beta}\alpha} + \frac{1}{2} D^{\alpha\beta}_n \mu^n_{\alpha\beta} + \frac{1}{2}\tilde{D}^{\dot{\alpha}\dot{\beta}}_{\bar{n}} \tilde{\mu}^{\bar{n}}_{\dot{\alpha}\dot{\beta}}.
\end{aligned} \tag{2.35}
$$

Upon integrating out $(\chi, \tilde{\chi}, s, \tilde{s}, D, \tilde{D})$, we obtain a Lagrangian which precisely coincides with the general Lagrangian (2.31) specialized to the abelian BF assignments of CS couplings and charge matrices. Note that, at first sight, the general Lagrangian (2.31) look different from (2.35) above, because (2.31) includes both unmixed and mixed couplings but (2.35) only allows mixed couplings. However, note that the $\mu^3/\tilde{\mu}^3$ parts of the bosonic potential in (2.31) vanish for abelian theories, and the $(\jmath\jmath)/(\tilde{\jmath}\tilde{\jmath})$ Yukawa couplings vanish when the k^{mn} is a BF-type.

Finally, we note that the two solutions to the constraints (2.33) are not mutually exclusive. In the next section, we will see some abelian theories which can be understood from both points of view up to a change of basis for the gauge fields.

2.3 Mass deformation

So far, we have restricted our attention to superconformal theories. In this subsection, we consider taking a mass deformation of the new theories. We will focus on mass parameters which preserve the SO(4) R-symmetry. In the GW construction procedure, the mass term is added to the superpotential as $W = W_0 + \Delta W + W_{\text{mass}}$, where W_0 and ΔW were defined in (2.26), (2.27), while W_{mass} is given by

$$
W_{\text{mass}} = \frac{m}{2}\epsilon^{\alpha\beta}\omega_{AB} q^A_\alpha q^B_\beta - \frac{m'}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\omega_{AB}\tilde{q}^A_{\dot{\alpha}}\tilde{q}^B_{\dot{\beta}}. \tag{2.36}
$$

Supersymmetry transformation rules should be modified accordingly, $\delta\Phi = \delta_0\Phi + \delta_{\text{mass}}\Phi$, for various fields Φ, where $\delta_0\Phi$ are the undeformed transformation (2.32). The only non-trivial changes due to the mass deformation are

$$
\delta_{\text{mass}}\psi^A_{\dot{\alpha}} = m q^A_\alpha \eta^\alpha{}_{\dot{\alpha}}, \qquad \delta_{\text{mass}}\tilde{\psi}^A_\alpha = m' \tilde{q}^A_{\dot{\alpha}} \eta_\alpha{}^{\dot{\alpha}}. \tag{2.37}
$$

A straightforward computation shows that the only additional contributions to the Lagrangian that can potentially break the SO(4) R-symmetry come from

$$
\begin{aligned}
\frac{1}{2}\left(F^A_\alpha\right)^2 &= \cdots - 2m\pi k_{mn}\epsilon^{\alpha\dot{\alpha}}\epsilon^{\beta\dot{\beta}}\mu^m_{\alpha\beta}\tilde{\mu}^n_{\dot{\alpha}\dot{\beta}}, \\
\frac{1}{2}\left(\tilde{F}^A_{\dot{m}}\right)^2 &= \cdots + 2m'\pi k_{mn}\epsilon^{\alpha\dot{\alpha}}\epsilon^{\beta\dot{\beta}}\mu^m_{\alpha\beta}\tilde{\mu}^n_{\dot{\alpha}\dot{\beta}}.
\end{aligned} \tag{2.38}
$$

[1] See section II of [50] for a review of the abelian BF theories in the $\mathcal{N} = 2$ superspace language.

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It follows that the two terms cancel against each other if and only if the two mass parameters for the hyper and twisted hyper-multiplets are equal: $m = m'$. In summary, the mass-deformed Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{mass}} = & -\omega_{AB}\left(\frac{m^2}{2}\epsilon^{\alpha\beta}q^A_\alpha q^B_\beta + \frac{m^2}{2}\epsilon^{\dot\alpha\dot\beta}\tilde q^A_{\dot\alpha}\tilde q^B_{\dot\beta} + \frac{i}{2}m\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}\psi^B_\beta - \frac{i}{2}m\epsilon^{\alpha\beta}\tilde\psi^A_\alpha\tilde\psi^B_\beta\right) \\ & -\frac{2\pi}{3}mk_{mn}\left((\mu^m)_{\alpha\beta}(\mu^n)^{\beta\alpha} - (\tilde\mu^m)_{\dot\alpha\dot\beta}(\tilde\mu^n)^{\dot\beta\dot\alpha}\right),\end{aligned} \tag{2.39}$$

with deformed SUSY variation rules

$$\delta_{\text{mass}}\psi^A_{\dot\alpha} = mq^A_\alpha\eta^\alpha{}_{\dot\alpha}, \qquad \delta_{\text{mass}}\tilde\psi^A_\alpha = m\tilde q^A_{\dot\alpha}\eta_\alpha{}^{\dot\alpha}. \tag{2.40}$$

3. Bagger-Lambert theory and M-crystal model

3.1 Bagger-Lambert theory

Bagger and Lambert [2] recently constructed three-dimensional $\mathcal{N} = 8$ superconformal Chern-Simons theories, believed to describe multiple M2-branes. In their construction, some 3-algebra with the four-index structure constant $f^{abc}{}_d$ was introduced. It can be however shown that the only possible 3-algebra is in fact SO(4) with $f^{abcd} = \varepsilon^{abcd}$, once we suppose $h^{ab} = \mathrm{tr}(t^a t^b)$ is positive definite [24, 25]. We will show that, from the $\mathcal{N} = 4$ perspective, the SO(4) BL model can be regarded as the GW model of the same gauge group with additional twisted hyper-multiplet.

BL theory. Let us start with a summary of our conventions for the BL theory. We use the mostly plus metric and mostly Hermitian eleven-dimensional Gamma matrices

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1), \qquad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.1}$$

Spinors Ψ^a and supersymmetry parameter ε are eigenvectors of Γ^{012} and $\Gamma^{3\cdots10}$ (we use $\Gamma^{012}\Gamma^{3\cdots10} = -1$) such that

$$\Gamma^{012}\Psi^a = -\Psi^a, \quad \Gamma^{012}\varepsilon = \varepsilon, \quad \Gamma^{3\cdots10}\Psi^a = \Psi^a, \quad \Gamma^{3\cdots10}\varepsilon = -\varepsilon. \tag{3.2}$$

Here a denote SO(4) gauge indices. The Lagrangian for the SO(4) BL model reads

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}D_\mu X^a_I D^\mu X^a_I + \frac{i}{2}\bar\Psi^a\Gamma^\mu D_\mu\Psi^a - \frac{i}{4\kappa}\varepsilon^{abcd}\,\bar\Psi^a X^b_I X^c_J\Gamma^{IJ}\Psi^d \\ & +\frac{\kappa\epsilon^{\mu\nu\lambda}}{2}\left(A^{ab}_\mu\partial_\nu\tilde A^{ab}_\lambda + \frac{2}{3}A^{ab}_\mu\tilde A^{ac}_\nu\tilde A^{bc}_\lambda\right) - \frac{1}{12\kappa^2}\sum_{IJK,a}(\varepsilon^{abcd}X^b_I X^c_J X^d_K)^2,\end{aligned} \tag{3.3}$$

and the supersymmetry transformation rules are

$$\begin{aligned}\delta X^a_I &= i\bar\varepsilon\Gamma_I\Psi^a, \\ \delta\Psi^a &= \not{D}X^a_I\Gamma^I\varepsilon + \frac{1}{6\kappa}\varepsilon^{abcd}X^b_I X^c_J X^d_K\Gamma^{IJK}\varepsilon, \\ \delta\tilde A^{ab}_\mu &= -\frac{i}{\kappa}\varepsilon^{abcd}\,\bar\varepsilon\Gamma_\mu\Gamma^I X^c_I\Psi^d.\end{aligned} \tag{3.4}$$

Hew we used the covariant derivatives and the tilded gauge field,

$$D\Psi^a \equiv d\Psi^a + \tilde{A}^{ab}\Psi^b, \qquad \tilde{A}^{ab} = \varepsilon^{abcd}A^{cd}.$$

It is noteworthy here that the standard quantization rule of Chern-Simons coupling gives $2\pi\kappa \in \mathbb{Z}$. In order to verify that this BL model nicely fits into the extended GW model with PSU(2|2), we first reduce the number of supersymmetry by half.

R-symmetry representation. In reducing $\mathcal{N} = 8$ supersymmetry to $\mathcal{N} = 4$, it is useful to keep track of how the SO(8) R-symmetry gets broken:

$$\mathrm{SO}(8) \supset \mathrm{SO}(4)_1 \times \mathrm{SO}(4)_2 \sim (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_A) \times (\mathrm{SU}(2)_B \times \mathrm{SU}(2)_R)\,. \tag{3.5}$$

The two SO(4) factors rotate $X_{3,4,5,6}$ and $X_{7,8,9,10}$ separately. For clarity, let us rename $X^a_{I=7,8,9,10}$ as Y^a_I from here on. In terms of the SU(2) factors, X_I transform as $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and Y_I as $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$. As for the spinors, define

$$\tilde{\psi} \equiv \frac{1+\Gamma^{3456}}{2}\Psi, \qquad \psi \equiv \frac{1-\Gamma^{3456}}{2}\Psi \qquad (\Gamma^{345678910}\Psi = \Psi), \tag{3.6}$$

$$\eta = \frac{1+\Gamma^{3456}}{2}\varepsilon, \qquad \tilde{\eta} = \frac{1-\Gamma^{3456}}{2}\varepsilon \qquad (\Gamma^{345678910}\varepsilon = -\varepsilon). \tag{3.7}$$

They transform under the four SU(2) factors as

$$\tilde{\psi} \,:\, (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}), \;\; \psi \,:\, (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}), \quad \eta \,:\, (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}), \;\; \tilde{\eta} \,:\, (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}). \tag{3.8}$$

Truncation from $\mathcal{N} = 8$ to $\mathcal{N} = 4$ amounts to setting $\tilde{\eta} = 0$. It is clear that, among the four SU(2) factors, $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ becomes the SO(4) R-symmetry of $\mathcal{N} = 4$. It is also clear from the supersymmetry variation rule ($X \sim \bar{\varepsilon}\Gamma\Psi$) that X_I and ψ form a hyper-multiplet q, and Y_I and $\tilde{\psi}$ form a twisted hyper-multiplet $\tilde{q}$.

As a short comment, one can truncate the $\mathcal{N} = 8$ BL model consistently down to $\mathcal{N} = 4$ by dropping the twisted hyper-multiplet: the truncated Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}D_\mu X^a_I D^\mu X^a_I + \frac{i}{2}\bar{\psi}^a\Gamma^\mu D_\mu\psi^a - \frac{i}{4\kappa}\varepsilon^{abcd}\bar{\psi}^a\Gamma_{IJ}\psi^b X^c_I X^d_J \\ &+\frac{\kappa\epsilon^{\mu\nu\lambda}}{2}\left(A^{ab}_\mu\partial_\nu\tilde{A}^{ab}_\lambda + \frac{2}{3}A^{ab}_\mu\tilde{A}^{ac}_\nu\tilde{A}^{bc}_\lambda\right) - \frac{1}{12\kappa^2}\sum_{IJK,a}(\varepsilon^{abcd}X^b_I X^c_J X^d_K)^2,\end{aligned} \tag{3.9}$$

where I here runs over 3 to 6. The truncated supersymmetry transformation rules are

$$\begin{aligned}\delta X^a_I &= i\bar{\eta}\Gamma_I\psi^a \\ \delta\psi^a &= \left(D_\mu X^a_I\Gamma^\mu\Gamma^I + \frac{1}{6\kappa}\varepsilon_{abcd}\Gamma^{IJK}X^b_I X^c_J X^d_K\right)\eta \\ \delta\tilde{A}^{ab}_\mu &= -\frac{i}{\kappa}\varepsilon^{abcd}\,\bar{\eta}\Gamma_\mu\Gamma^I X^c_I\psi^d.\end{aligned} \tag{3.10}$$

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Explicit embedding. For explicit comparison, we will use the doublet indices

$$(\alpha, \beta; \dot{\sigma}, \dot{\tau}; \sigma, \tau; \dot{\alpha}, \dot{\beta})$$

for the four SU(2) R-symmetry factors. For each SU(2), indices are raised and lowered by the invariant anti-symmetric tensor satisfying $\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta$. The pseudo-reality condition for a "spinor" reads

$$(u_\alpha)^\dagger = \bar{u}^\alpha = \epsilon^{\alpha\beta} u_\beta.$$

Explicit embedding of the SU(2) factors into the SO(8) is facilitated by a specific basis for the eleven-dimensional gamma matrices,

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes (-\gamma^5) \otimes \gamma^5 && \text{for } \mu = 0, 1, 2 \\ \Gamma^I &= \mathbf{1}_2 \otimes \gamma^I \otimes \mathbf{1}_4 && \text{for } I = 3, 4, 5, 6 \\ \Gamma^J &= \mathbf{1}_2 \otimes \gamma^5 \otimes \tilde{\gamma}^J && \text{for } J = 7, 8, 9, 10 \end{aligned} \tag{3.11}$$

with hermitian SO(4) gamma matrices

$$\gamma^I = \begin{pmatrix} 0 & (e^I)_{\alpha\dot{\tau}} \\ (\bar{e}^I)^{\dot{\sigma}\beta} & 0 \end{pmatrix}, \tag{3.12}$$

and the real three-dimensional gamma matrices γ^μ. These γ^μ are chosen to satisfy $\gamma^{012} = 1$. Here $e^I = (i\vec{\sigma}, 1)$ and $\bar{e}^I = (-i\vec{\sigma}, 1)$ satisfy the reality condition

$$(e^I_{\alpha\dot{\sigma}})^* = \epsilon^{\alpha\beta}(e^I)_{\beta\dot{\tau}}\epsilon^{\dot{\tau}\dot{\sigma}}. \tag{3.13}$$

The other four gamma matrices $\tilde{\gamma}^J$ are numerically identical to γ^I, but of course carry different SU(2) indices. In this basis, the reality condition (Majorana condition) on fermion fields Ψ is given by

$$\Psi^* = B\Psi, \qquad B = \Gamma^{3579} = \mathbf{1}_2 \otimes C \otimes C, \tag{3.14}$$

where C denote the charge-conjugation operator

$$C = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}. \tag{3.15}$$

Decomposing the spinors as

$$\Psi \;\rightarrow\; \tilde{\psi}_{\alpha\sigma} \oplus \psi^{\dot{\sigma}\dot{\alpha}}, \qquad \varepsilon \;\rightarrow\; \eta_\alpha{}^{\dot{\alpha}} \oplus \tilde{\eta}^{\dot{\sigma}}{}_\sigma, \tag{3.16}$$

one can show that the reality conditions (3.14) become

$$(\tilde{\psi}_{\alpha\sigma})^* = \epsilon^{\alpha\beta}\epsilon^{\sigma\tau}\tilde{\psi}_{\beta\tau}, \;\; \text{etc.} \tag{3.17}$$

For later convenience, let us re-express the scalars as bi-spinors via

$$X_I \;\rightarrow\; \bar{X}^{\dot{\sigma}\alpha} \equiv \frac{1}{2} X_I (\bar{e}^I)^{\dot{\sigma}\alpha}, \;\; \text{etc. ,} \tag{3.18}$$

whose reality condition can be read off from (3.13). After all the replacements, the Lagrangian can now be described as

$$\begin{aligned}\mathcal{L} = &\operatorname{tr}(-D_\mu \bar{X}^a D^\mu X^a - D_\mu \bar{Y}^a D^\mu Y^a + \frac{i}{2}\psi^a \gamma^\mu D_\mu \psi^a + \frac{i}{2}\tilde{\psi}^a \gamma^\mu D_\mu \tilde{\psi}^a) \\ &-\frac{i}{\kappa}\varepsilon^{abcd}\left(\psi^a{}_{\dot\sigma\dot\alpha}(\bar{X}^c X^d)^{\dot\sigma}{}_{\dot\tau}\psi^{b\dot\tau\dot\alpha} + \tilde{\psi}^{a\alpha\sigma}(Y^c\bar{Y}^d)_\sigma{}^\tau\tilde{\psi}^b{}_{\alpha\tau} - 4\psi^a{}_{\dot\sigma\dot\alpha}\bar{X}^{c\dot\sigma\alpha}\bar{Y}^{d\dot\alpha\sigma}\tilde{\psi}^b_{\alpha\sigma}\right) \\ &+\frac{\kappa\epsilon^{\mu\nu\lambda}}{2}\left(A^{ab}_\mu\partial_\nu\tilde{A}^{ab}_\lambda + \frac{2}{3}A^{ab}_\mu\tilde{A}^{ac}_\nu\tilde{A}^{bc}_\lambda\right) - V(X,Y),\end{aligned} \tag{3.19}$$

with scalar potential $V = V_1(X) + V_1(Y) + V_2(X,Y) + V_2(Y,X)$ with

$$\begin{aligned} V_1(X) &= -\frac{4}{9\kappa^2}\varepsilon_{abcd}\varepsilon_{aefg}\operatorname{tr}\left(X^b\bar{X}^c X^d\bar{X}^e X^f\bar{X}^g\right), \\ V_2(X,Y) &= -\frac{2}{\kappa^2}\varepsilon_{abcd}\varepsilon_{aefg}\operatorname{tr}\left(X^b\bar{X}^e\right)\operatorname{tr}\left(Y^c\bar{Y}^d Y^f\bar{Y}^g\right).\end{aligned} \tag{3.20}$$

The supersymmetry transformation rules (3.4) can be recast as

$$\begin{aligned} \delta X^a_{\alpha\dot\sigma} &= i\eta_\alpha{}^{\dot\alpha}\psi^a{}_{\dot\sigma\dot\alpha}, \qquad \delta Y^a_{\sigma\dot\alpha} \;=\; -i\tilde{\psi}^a{}_{\alpha\sigma}\eta^\alpha{}_{\dot\alpha}, \\ \delta\psi^a{}_{\dot\sigma\dot\alpha} &= \left[-2\gamma^\mu D_\mu X^a{}_{\alpha\dot\sigma} + \frac{8\pi}{3n}\varepsilon_{abcd}(X^b\bar{X}^c X^d)_{\alpha\dot\sigma}\right]\eta^\alpha{}_{\dot\alpha} + \frac{8\pi}{n}\varepsilon_{abcd}X^b{}_{\alpha\dot\sigma}\eta^\alpha{}_{\dot\beta}(\bar{Y}^c Y^d)^{\dot\beta}{}_{\dot\alpha}, \\ \delta\tilde{\psi}^a{}_{\alpha\sigma} &= \left[2\gamma^\mu D_\mu Y^a_{\sigma\dot\alpha} + \frac{8\pi}{3n}\varepsilon^{abcd}(Y^b\bar{Y}^c Y^d)_{\sigma\dot\alpha}\right]\eta_\alpha{}^{\dot\alpha} + \frac{8\pi}{n}\varepsilon^{abcd}Y^b_{\sigma\dot\alpha}(X^c\bar{X}^d)_\alpha{}^\beta\eta_\beta{}^{\dot\alpha}, \\ \delta\tilde{A}^{ab}_\mu &= -\frac{4\pi i}{n}\varepsilon_{abcd}\eta^\alpha{}_{\dot\alpha}\gamma_\mu\left(\psi^{d\dot\sigma\dot\alpha}X^c{}_{\alpha\dot\sigma} + \tilde{\psi}^d{}_{\alpha\sigma}\bar{Y}^{c\dot\alpha\sigma}\right).\end{aligned} \tag{3.21}$$

where we used the fact that κ is quantized as $2\pi\kappa = n$ (n is integer). Before the detailed comparison, let us first present the convenient choice of symplectic embedding of $\mathrm{SO}(4)_{\mathrm{gauge}}$.

Symplectic embedding. For the PSU(2|2) GW theory, the matters transform as bi-fundamentals under the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$, or as vector under $\mathrm{SO}(4)_{\mathrm{gauge}}$. For clarity, we will use the latter convention throughout this section. The representation under $\mathrm{SO}(4)_{\mathrm{gauge}}\times\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ symmetry, aside from the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry, of the BL model can be related to the symplectic notation of Gaiotto and Witten as follows: a symplectic index A on hyper-multiplets can be decomposed into $(a,\dot\sigma)$ while B on twisted ones can be decomposed into (b,σ). Here a,b denote the $\mathrm{SO}(4)_{\mathrm{gauge}}$ index, and $\dot\sigma$ and σ transform as doublets under $\mathrm{SU}(2)_A$ and $\mathrm{SU}(2)_B$, respectively.

Under this symplectic embedding of $\mathrm{SO}(4)_{\mathrm{gauge}}$ into Sp(8), the symplectic indices simplifies the index structure of various matter fields as

$$\begin{aligned} q^A_\alpha &= \sqrt{2}X^a{}_{\alpha\dot\sigma}, \qquad \tilde{q}^A_{\dot\alpha} \;=\; \sqrt{2}Y^a{}_{\sigma\dot\alpha}, \\ \psi^A_{\dot\alpha} &= \psi^a{}_{\dot\sigma\dot\alpha} \qquad\quad \tilde{\psi}^A_\alpha \;=\; \tilde{\psi}^a{}_{\alpha\sigma}\end{aligned} \tag{3.22}$$

The symplectic invariant tensor and the gauge generators are then decomposed as

$$\omega_{AB} = \delta_{ab}\otimes\epsilon^{\dot\sigma\dot\tau}, \qquad t^m_{AB} = t^m_{ab}\otimes\epsilon^{\dot\sigma\dot\tau}, \tag{3.23}$$

with $\mathrm{SO}(4)_{\mathrm{gauge}}$ group generators t^m_{ab}. Clearly, the same argument goes through for the twisted hyper-multiplets:

$$\tilde{\omega}_{AB} = \delta_{ab} \otimes \epsilon^{\sigma\tau}. \qquad \tilde{t}^m_{AB} = t^m_{ab} \otimes \epsilon^{\sigma\tau}. \tag{3.24}$$

We are now ready to show the equivalence between the BL model and the extend GW model with the supergroup PSU(2|2).

Equivalence. Re-normalizing the supersymmetry parameters η as $\eta_\alpha{}^{\dot\alpha} \to \frac{1}{\sqrt{2}}\eta_\alpha{}^{\dot\alpha}$, together with the choice of symplectic embedding above, one can obtain from (3.21) the following supersymmetry variation rules

$$\begin{aligned}
\delta q^A_\alpha &= i\eta_\alpha{}^{\dot\alpha}\psi^A_{\dot\alpha}, \\
\tilde{q}^A_{\dot\alpha} &= -i\tilde{\psi}^A_\alpha \eta^\alpha{}_{\dot\alpha}, \\
\delta\psi^A{}_{\dot\alpha} &= \left[\gamma^\mu D_\mu q^A_\alpha + \frac{2\pi}{3}k_{mn}(t^m)^A{}_B q^B_\beta (\mu^n)^\beta{}_\alpha\right]\eta^\alpha{}_{\dot\alpha} - 2\pi k_{mn}(t^m)^A{}_B q^B_\alpha (\tilde{\mu}^n)_{\dot\alpha}{}^{\dot\beta}\eta^\alpha{}_{\dot\beta} \\
\delta\tilde{\psi}^A{}_\alpha &= -\left[\gamma^\mu D_\mu q^A_\alpha + \frac{2\pi}{3}k_{mn}(\tilde{t}^m)^A{}_B \tilde{q}^B_{\dot\beta}(\tilde{\mu}^n)^{\dot\beta}{}_{\dot\alpha}\right]\eta_\alpha{}^{\dot\alpha} + 2\pi k_{mn}(\tilde{t}^m)^A{}_B \tilde{q}^B_{\dot\alpha}(\mu^n)^\beta{}_\alpha \eta_\beta{}^{\dot\alpha} \\
\delta\tilde{A}^m_\mu &= 2\pi i\eta^{\alpha\dot\alpha}\rho_\mu(q^A_\alpha \tau^m_{AB}\psi^B_{\dot\alpha} - \tilde{q}^A_{\dot\alpha}\tilde{\tau}^m_{AB}\tilde{\psi}^B_\alpha),
\end{aligned} \tag{3.25}$$

where $\tilde{A}_\mu$ is defined by $\tilde{A}^{ab}_\mu = k_{mn}(t^m)_{ab}\tilde{A}^n_\mu$ and we chose a basis for $\mathrm{SO}(4)_{\mathrm{gauge}}$ such that

$$(t^m)_{ab} = (\vec{t}_L, \vec{t}_R)_{ab}, \qquad k^{mn} = k\ \mathrm{diag}(\mathbf{1}_3, -\mathbf{1}_3), \tag{3.26}$$

where $\vec{t}_L$ and $\vec{t}_R$ denote self-dual and anti-self-dual $\mathrm{SO}(4)_{\mathrm{gauge}}$ generators. Here k now is quantized to be an even integer. A useful identity in our discussion is

$$k_{mn}(t^m)_{ab}(t^n)_{cd} = \frac{2}{k}\varepsilon_{abcd}. \tag{3.27}$$

The transformation rules (3.25) are in perfect agreement with those in (2.32) of the extended GW model in section 2, which strongly implies the equivalence between the two models. One can also easily show that the Lagrangian (3.19) reduces to (2.31).

In summary, we conclude that the BL model in the $\mathcal{N} = 4$ notation is identical to the extended GW model with the supergroup PSU(2|2).

Mass deformation. We now show that the mass-deformed BL model proposed in [21, 23] is in fact a special example of the mass-deformed extended GW model in section 2.3. The Lagrangian of the mass-deformed BL model can be written as $\mathcal{L} = \mathcal{L}_{\mathrm{BL}} + \mathcal{L}_{\mathrm{mass}}$ with

$$\mathcal{L}_{\mathrm{mass}} = -\frac{m^2}{2}(X^a_I)^2 - \frac{i}{2}m\bar{\Psi}^a\Gamma_{3456}\Psi^a + \frac{4m}{\kappa}\varepsilon^{abcd}(X^a_3X^b_4X^c_5X^d_6 + X^a_7X^b_8X^c_9X^d_{10}), \tag{3.28}$$

which is invariant under the deformed supersymmetry transformation rules, $\delta\Phi = \delta_{\mathrm{BL}}\Phi + \delta_{\mathrm{mass}}\Phi$, for various fields Φ with

$$\delta_{\mathrm{mass}}\Psi^a = m\Gamma_{3456}X^a_I\Gamma^I\epsilon. \tag{3.29}$$

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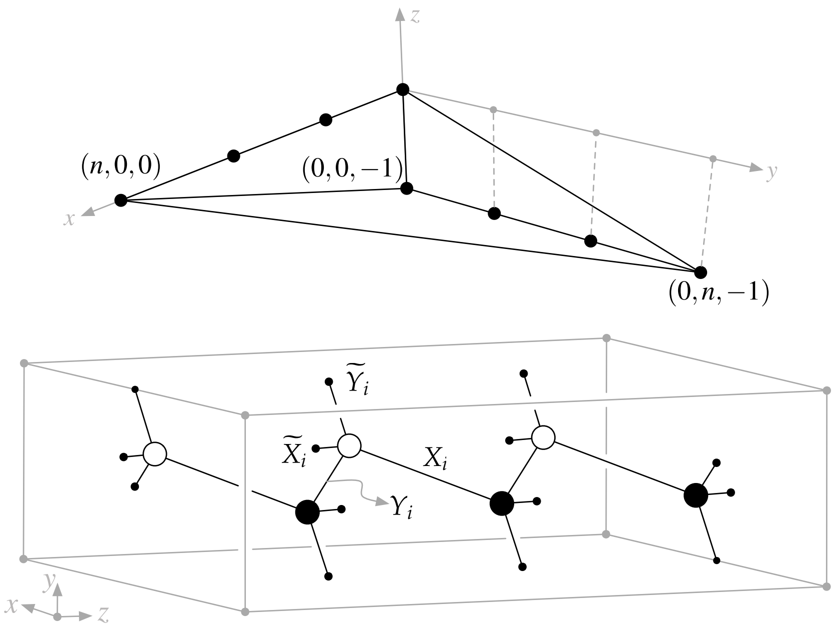

Figure 2: Toric diagram and crystal for $(\mathbb{C}^2/\mathbb{Z}_n)^2$ (reproduced from [43]).

This deformation preserves the maximal, say sixteen real, supersymmetries. Let us rewrite them in the $\mathcal{N} = 4$ language. Based on the specific representation of eleven-dimensional gamma matrices together with the choice of symplectic embedding in this section, the deformed supersymmetry variation rules now becomes

$$\begin{aligned} \delta_{\text{mass}}\psi^{a}{}_{\dot\sigma\dot\alpha} &= 2mX^{a}{}_{\alpha\dot\alpha}\eta^{\alpha}{}_{\dot\alpha} \;\;\rightarrow\;\; \delta_{\text{mass}}\psi^{A}_{\dot\alpha} = mq^{A}_{\alpha}\eta^{a}{}_{\dot\alpha} \\ \delta_{\text{mass}}\tilde\psi^{a}{}_{\alpha\sigma} &= 2mY^{a}{}_{\sigma\dot\alpha}\eta_{\alpha}{}^{\dot\alpha} \;\;\rightarrow\;\; \delta_{\text{mass}}\tilde\psi^{A}_{\alpha} = m\tilde q^{A}_{\dot\alpha}\eta_{\alpha}{}^{\dot\alpha}, \end{aligned} \tag{3.30}$$

which agrees with (2.40). It implies the equivalence between the two models. One can also show that the interaction terms in (3.28) reduce to those in (2.39).

3.2 Abelian quiver and M-crystal model

Review of the M-crystal. The M-crystal model for M2-branes probing the $(\mathbb{C}^2/\mathbb{Z}_n)^2$ orbifold was studied in [43]. The toric diagram of the orbifold and the associated crystal diagram are reproduced in figure 2.

In the crystal diagram, the "bonds" correspond to the $\mathcal{N} = 2$ holomorphic matter super-fields. The "atoms" of the crystal encode the super-potential; take the products of all fields ending on a given atom and sum over all atoms with opposite signs between the white and black atoms. The result is

$$W = \sum_{i=1}^{n} \left(X_i\widetilde{X}_iY_i\widetilde{Y}_i - X_i\widetilde{X}_iY_{i+1}\widetilde{Y}_{i+1} \right), \tag{3.31}$$

from which we find the F-term conditions,

$$X_i\widetilde{X}_i - X_{i-1}\widetilde{X}_{i-1} = 0, \qquad Y_i\widetilde{Y}_i - Y_{i+1}\widetilde{Y}_{i+1} = 0. \tag{3.32}$$

	X_1	$\tilde{X}_1$	X_2	$\tilde{X}_2$	Y_1	$\tilde{Y}_1$	Y_2	$\tilde{Y}_2$
Q^X	$-$	$+$	$-$	$+$	0	0	0	0
$\tilde{Q}^X$	$+$	$-$	$+$	$-$	0	0	0	0
Q^Y	0	0	0	0	$+$	$-$	$+$	$-$
$\tilde{Q}^Y$	0	0	0	0	$-$	$+$	$-$	$+$

Table 1: Charge assignments of $(C^2/Z_2)^2$ orbifold theory

In a more delicate but systematic way explained in [43], the crystal diagram also determines the charges of the abelian probe theory. In the case at hand, the gauge group turns out to be $(\mathrm{U}(1)^n/\,\mathrm{U}(1)_D)_X \times (\mathrm{U}(1)^n/\,\mathrm{U}(1)_D)_Y$. The matter fields $(X_i, \widetilde{X}_i)$ are charged only under the first factor, $(\mathrm{U}(1)^n/\,\mathrm{U}(1)_D)_X$, as

	X_i	$\widetilde{X}_i$
Q_i	$+$	$-$
Q_{i+1}	$-$	$+$

$$(i = 1, \cdots, n)$$

with all other charges vanishing. The matter fields $(Y_i, \widetilde{Y}_i)$ are charged under the second factor, $(\mathrm{U}(1)^n/\,\mathrm{U}(1)_D)_Y$, in the same way.

The proposal for the abelian theory in [43] was incomplete as the gauge kinetic terms were not specified. It was assumed that the D-term potential exists and effectively complexifies the gauge group, as is commonly true of gauge theories with four supercharges. Under these assumptions, the moduli space of vacua was shown to coincide with the orbifold $(\mathbb{C}^2/\mathbb{Z}_n)^2$. In fact, the orbifold theories have $\mathcal{N} = 4$ supersymmetry with the $\mathcal{N} = 2$ fields paring up to form (twisted) hyper-multiplets,

$$(X_i, \widetilde{X}_i^\dagger) \to q_i, \quad (Y_i, \widetilde{Y}_i^\dagger) \to \tilde{q}_i, \tag{3.33}$$

so the F-term and D-term conditions together are expected to result in a hyper-Kähler quotient [44] for the moduli space of vacua.

We illustrate the identification of the moduli space of vacua using the simplest $(\mathbb{C}^2/\mathbb{Z}_2)^2$ orbifold. The theory has two hyper-multiplets $(X_i, \tilde{X}_i)$ $(i = 1, 2)$ and two twisted hyper-multiplets $(Y_i, \tilde{Y}_i)$. Their charges are summarized in the table 1, and the superpotential is given by

$$W = (X_1\tilde{X}_1 - X_2\tilde{X}_2)(Y_1\tilde{Y}_1 - Y_2\tilde{Y}_2). \tag{3.34}$$

The F-term conditions require that $X_1\tilde{X}_1 = X_2\tilde{X}_2 \equiv u_3$ and $Y_1\tilde{Y}_1 = Y_2\tilde{Y}_2 \equiv v_3$. The gauge invariant monomials $u_1 = X_1\tilde{X}_2$, $u_2 = \tilde{X}_1X_2$, $v_1 = Y_1\tilde{Y}_2$ and $v_2 = \tilde{Y}_1Y_2$ together with u_3 and v_3 parameterize the moduli space of vacua. Based on the F-term conditions, we find

$$u_1u_2 = u_3^2, \qquad v_1v_2 = v_3^2, \tag{3.35}$$

which are nothing but the algebraic descriptions for each of the two $(\mathbb{C}^2/\mathbb{Z}_2)$ factors.

In ref. [43], it was not known how the abelian theory can be completed in a manifestly $\mathcal{N} = 4$ manner. We will show that a particular abelian quiver theory of the present paper satisfies all the properties of such a completion.

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The connection. Consider the abelian $\hat{A}_{2n-1}$ quiver theory, which has $2n$ nodes connected to make a circle. We assign

- A U(1) gauge field A_m $(m = 1, \cdots, 2n)$ to each node.
- A hyper-multiplet q_i to each of the n links $\langle 2i-1, 2i\rangle$ $(i = 1, \cdots, n)$.
- A twisted hyper-multiplet $\tilde{q}_i$ to each of the n links $\langle 2i, 2i+1\rangle$ $(i = 1, \cdots, n)$.

The hyper and twisted hyper-multiplets give two ways to uplift the group $\mathrm{U}(1)^{2n}$ to $\mathrm{U}(1|1)^n$. The quiver diagram encodes the charges of the matter fields. The GW construction requires the Chern-Simons coupling to be

$$k^{mn} = k\,\mathrm{diag}\,\underbrace{(1, -1, \cdots, 1, -1)}_{2n}.$$

For this model, the scalar potential as specified in (2.31) can be simplified as

$$V = 2\pi^2 \sum_{i=1}^{n} \left[|\tilde{q}_i|^2 (\mu^{2i-1} - \mu^{2i+1})^2 + |q_i|^2 (\tilde{\mu}^{2i-2} - \tilde{\mu}^{2i})^2 \right], \tag{3.36}$$

where the squares of the moment maps are defined by $(\mu)^2 \equiv \mu_{\alpha\beta}\mu^{\alpha\beta}$ and $\tilde{\mu}^2 \equiv \tilde{\mu}_{\dot{\alpha}\dot{\beta}}\tilde{\mu}^{\dot{\alpha}\dot{\beta}}$. The vacuum conditions from the scalar potential become

$$\mu^1 = \mu^3 = \cdots = \mu^{2n-1}, \qquad \tilde{\mu}^2 = \tilde{\mu}^4 = \cdots = \tilde{\mu}^{2n}, \tag{3.37}$$

which coincide with the $\mathcal{N} = 4$ covariant version of the F-term conditions (3.32).

To compare the gauge symmetries here with those of the M-crystal model, we make the following change of basis for the charges,

$$\begin{aligned} Q^+ &= \sum_{m=1}^{2n} Q^m, & Q^- &= \frac{1}{2}\sum_{i=1}^{n}(Q^{2i-1} - Q^{2i}), \\ \hat{Q}^i &= Q^{2i-2} + Q^{2i-1} - \frac{1}{n}Q^+, & \check{Q}^i &= Q^{2i-1} + Q^{2i} - \frac{1}{n}Q^+, \end{aligned} \tag{3.38}$$

and define a new set of U(1) gauge fields

$$\sum_{m=1}^{2n} A_m Q^m = A_+ Q^+ + A_- Q^- + \sum_j \hat{A}_j \hat{Q}^j + \sum_j \check{A}_j \check{Q}^j. \tag{3.39}$$

Recalling our charge assignments

$$Q^{2i-1}[q_i] = 1, \quad Q^{2i}[q_i] = -1, \quad Q^{2i}[\tilde{q}_i] = 1, \quad Q^{2i+1}[\tilde{q}_i] = -1,$$

we immediately see that q_i are charged under $\hat{Q}^j$ in the same way as in the crystal model and neutral under $\check{Q}^j$. Similarly, $\tilde{q}_i$ are charged under $\check{Q}^j$ and neutral under $\hat{Q}^j$. Note that the number of independent $\hat{Q}^j$ $(\check{Q}^j)$ are $(n-1)$ as in the crystal model. One can also show from the vacuum conditions (3.37) that the moment maps associated to $\hat{A}_i, \check{A}_i$ gauge groups should vanish

$$\check{\mu}^i_{\alpha\beta} = 0 \qquad \hat{\mu}^i_{\dot{\alpha}\dot{\beta}} = 0 \quad . \tag{3.40}$$

All matter fields are neutral under A_+, so A_+ decouple from the dynamics of matter fields. Under Q^-, all q_i have charge $+1$, and all $\tilde{q}_i$ have -1. As for the Chern-Simons coefficient, there are BF type couplings between $\hat{A}_i$ and $\check{A}_j$, another BF coupling between A_- and A_+, and all other couplings vanish:

$$\mathcal{L}_{\rm CS} = \frac{2nk}{4\pi}\epsilon^{\mu\nu\rho}A_{-\mu}\wedge\partial_\nu A_{+\rho} + \cdots\,, \tag{3.41}$$

where we are only keeping the BF-coupling of our interest in the following discussion.[2]

Aside from the fact that the $\mathrm{U}(1)_{Q^-}$ symmetry is gauged, the vacuum conditions (3.40) together with gauge symmetries of $(\hat{A}_i,\ \check{A}_i)$ can be understood as the standard hyper-Kähler quotient construction for the orbifold $(\mathbb{C}^2/\mathbb{Z}_n)^2$. If we further mod it out by the gauge symmetry of Q^-, it appears that we are left with a seven-dimensional moduli space, in contradiction to the well-established link between $\mathcal{N}=4$ supersymmetry and hyper-Kähler spaces.

A similar problem was encountered in [12, 13], where there was also a problematic $\mathrm{U}(1)$ gauge field (call it A_-) and a decoupled $\mathrm{U}(1)$ gauge field (call it A_+) coupled to each other through a BF coupling. The naive result was a strange 15-dimensional moduli space of vacua. The authors of [12] resolved the problem by dualizing A_+ to a scalar σ and fixing the problematic A_- gauge symmetry by setting $\partial\sigma \sim A_-$. The idea can be carried over to our current setup.

Starting with the Lagrangian for the present abelian quiver theories

$$\mathcal{L}_{\rm quiver} = -\frac{1}{2}(|D_\mu q_i|^2 + |D_\mu \tilde{q}_i|^2) + \frac{nk}{2\pi}\epsilon^{\mu\nu\rho}A_{-\mu}\wedge\partial_\nu A_{+\rho} + \cdots - V(q_i,\tilde{q}_i), \tag{3.42}$$

we replace the decoupled $\mathrm{U}(1)$ gauge field A_+ by its dual scalar σ through introducing the well-known Lagrange multiplier

$$\mathcal{L}_{\rm dual} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\sigma\partial_\mu F_{+\nu\rho}. \tag{3.43}$$

(If we allow nonabelian embedding, magnetic monopoles would be present and the variable σ would be periodic.) Here the covariant derivatives of matters are

$$D_\mu q_i = (\partial_\mu - iA_{-\mu} + \cdots)q_i, \qquad D_\mu\tilde{q}_i = (\partial_\mu + iA_{-\mu} + \cdots)\tilde{q}_i, \quad \text{etc.}, \tag{3.44}$$

where we suppressed all terms irrelevant for our discussions. It is noteworthy again that the decoupled A^+_μ enters into the theories through the BF-coupling with A^-_μ. Collecting all interactions, one finally obtain

$$\mathcal{L}_{\rm tot} = -\frac{1}{2}(|D_\mu q_i|^2 + |D_\mu\tilde{q}_i|^2) + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}(nkA_{-\mu} - \partial_\mu\sigma)F_{+\nu\rho} + \cdots\ . \tag{3.45}$$

[2] ref. [34] discusses orbifolding the SO(4) BL theory to obtain an $\mathrm{U}(1)^2$ gauge theory, which resembles our $(\mathbb{C}^2/\mathbb{Z}_2)^2$ theory.

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Note that $\mathcal{L}_{\text{tot}}$ is invariant under the gauge transformation generated by Q^-, which is

$$q_i \to e^{i\theta} q_i, \quad \sigma \to \sigma + nk\theta, \quad A_- \to A_- + d\theta. \tag{3.46}$$

One can fix the problematic gauge field $A_{-\mu}$ by eliminating the Lagrange multiplier $F_{+\nu\rho}$

$$A_{-\mu} = \frac{1}{nk}\partial_\mu \sigma. \tag{3.47}$$

In terms of the local gauge invariant fields $e^{i\sigma/nk}q_i$, one could have a possibility of further discrete identification depending on k if σ is periodic.

To conclude, with the field redefinition $q_i \to e^{-\frac{i}{nk}\sigma} q_i$, the theory effectively decouples from A_- and recovers all the desired properties of the abelian M-crystal model discussed earlier, except the possible discrete identification.

4. Discussion

We have constructed a large class of new three-dimensional $\mathcal{N} = 4$ superconformal Chern-Simons field theories with both types of hyper-multiplets and their mass deformations, which are generically a linear quiver theory. We find that the BL theory and the M-crystal models fit well with our theories. It remains to study physical properties of these theories such as their moduli space of vacua and BPS states. We leave them for future works. There are a few other directions which deserve further study.

The Gaiotto-Witten construction and its extension of the present paper probably exhaust all possible $\mathcal{N} = 4$ superconformal Chern-Simons theories minimally coupled to (twisted) hyper-multiplets. In ref. [6], the construction was generalized to a sigma model whose target space is any hyper-Kähler manifold. We expect our extended construction will also allow such generalization.

A new superconformal family of BL models as a possible theory on $M2$ branes were constructed quite recently where the positive definiteness of the group space is relaxed and so the 3-algebra structure can be associated with arbitrary ordinary Lie algebras [29–31]. This theory has unconventional BF couplings and the Gaiotto-Witten's work and our extension can be explored along this direction too. It remains to be seen how useful this further extension, which does not fall into the classification of this paper, would be.

In the original GW construction [6], the appearance of $\mathrm{U}(N|M)$ or $\mathrm{OSp}(N|M)$ theories were related to of D3(O3)-branes intersecting with NS5-branes in IIB string theory. It is natural to ask whether similar interpretation is possible for our present work which includes twisted hyper-multiplets. To see this, let us explore the following alignments of branes:

	0	1	2	3	4	5	6	7	8	9
D3(O3)	○	○	○	○						
NS5	○	○	○		○	○	○			
D5	○	○	○					○	○	○

Each of D3- and NS5-branes breaks half of supersymmetries, leading to eight real supersymmetries. Additional D5-branes introduced above do not break supersymmetry any

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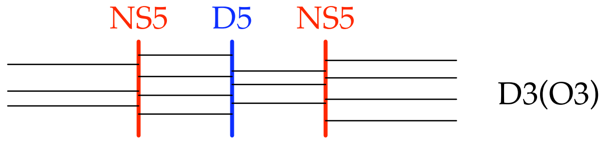

Figure 3: Brane configuration for the extended GW construction.

further. In the above brane setting, the $SU(2)_L \times SU(2)_R$ R-symmetry is realized as $SO(3)_{456} \times SO(3)_{789}$. Since the hyper-multiplet in the original Gaiotto-Witten construction is associated to the presence of NS5-branes, D5 branes would lead to the appearance of twisted hyper-multiplets. If such a suggestion works out correctly, the quiver diagram of figure 1 would correspond to the following brane configuration.

The above argument suggests that our quiver theory have a type IIB origin. As our theory seems to describe the physics on M2 branes on the tip of the $(\mathbb{C}^2/\mathbb{Z}_n)^2$ orbifold, we want to find some duality transformation between the two approaches. Now, replace each $\mathbb{C}^2/\mathbb{Z}_n$ by the singular Taub-NUT space and keep the radius of the Kaluza-Klein circles of the Taub-NUT space to be much larger than the 11-dimensional Plank length. Then T-dualize along two circles to get IIB string theory on a circle. Under this duality, M2-branes turn into D3-branes and the two Taub-NUT spaces turn into NS5/D5-branes. Therefore, we obtain precisely the IIB brane configuration discussed above, except that the x^3-direction is compactified. This duality transformation seems to be a right step toward relating the M-theoretic aspect of our quiver theories to their type IIB interpretation.

Acknowledgments

We thank Andreas Karch, Jun Nishimura and Piljin Yi for discussions. Sm.L. is grateful to the organizers of the INT workshop “From Strings to Things” at the University of Washington in Seattle where part of the work was done. K.M.L. and J.P. are supported in part by the KOSEF SRC Program through CQUeST at Sogang University. K.M.L. is supported in part by KRF Grant No. KRF-2005-070-C00030, and the KRF National Scholar program. Sm.L. is supported in part by the KOSEF Grant R01-2006-000-10965-0 and the Korea Research Foundation Grant KRF-2007-331-C00073. J.P. is supported in part by the Stanford Institute for Theoretical Physics.

A. Notations and conventions

Spinor calculus. Spinor indices run $\alpha = +,-$. Indices are raised or lowered by real antisymmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ satisfying $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$.

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta.$$

Space-time metric has signature $(-++)$. The γ-matrices $(\gamma^\mu)_\alpha^{\ \beta}$ satisfy the relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \qquad \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \varepsilon^{\mu\nu\rho}. \qquad (\varepsilon^{012} = 1)$$

The matrices $(\epsilon\gamma^\mu)^{\alpha\beta}$ are real symmetric. Vectors such as x^μ and ∂_μ are expressed as bi-spinors

$$x^{\alpha\beta} = x^\mu(\epsilon\gamma_\mu)^{\alpha\beta}, \qquad \partial_{\alpha\beta} = -(\gamma^\mu\epsilon)_{\alpha\beta}\partial_\mu.$$

Spinor indices in the standard position will be omitted.

$$\psi\theta \equiv \psi^\alpha\theta_\alpha, \qquad \theta^2 = \theta^\alpha\theta_\alpha, \qquad \psi\gamma^\mu\theta = \psi^\alpha(\gamma^\mu)_\alpha^{\ \beta}\theta_\beta, \qquad \text{etc.}$$

Superspace and superfields. $\mathcal{N} = 1$ superspace coordinates are $x^{\alpha\beta}$ and θ^α (both real). Supersymmetry algebra is realized in terms of supertranslations

$$P_{\alpha\beta} = i\partial_{\alpha\beta}, \quad Q_\alpha = i\partial_\alpha + \theta^\beta\partial_{\beta\alpha}.$$

Supercovariant derivatives are defined by

$$D_{\alpha\beta} = \partial_{\alpha\beta}, \qquad D_\alpha = \partial_\alpha + i\theta^\beta\partial_{\beta\alpha}. \qquad \{D_\alpha, D_\beta\} = 2iD_{\alpha\beta}. \tag{A.1}$$

Components of general superfields Φ are defined by $D_\alpha\cdots D_\beta\Phi|$, where $|$ takes the zeroth term of the power series in θ. Supersymmetry acts on superfields as superderivatives. Its action on components $(\cdots)|$ can be written as

$$\delta_\xi(\cdots)| \;\equiv\; -i\xi^\alpha(Q_\alpha\cdots)|.$$

For example, scalar superfield Φ takes the form

$$\Phi = \phi + i\theta\psi - \frac{i}{2}\theta^2F, \quad \phi = \Phi|, \quad \psi_\alpha = -iD_\alpha\Phi|, \quad F = -\frac{i}{2}D^2\Phi|, \tag{A.2}$$

where $D^2 \equiv D^\alpha D_\alpha$. Supersymmetry transformation law of components becomes

$$\delta\phi = i\xi^\alpha\psi_\alpha, \qquad \delta\psi_\alpha = -\partial_\alpha^{\ \beta}\phi\xi_\beta - F\xi_\alpha, \qquad \delta F = i\xi^\beta\partial_\beta^{\ \alpha}\psi_\alpha. \tag{A.3}$$

Gauge symmetry is described by superconnections on the superspace. As a simple example, consider a column vector Φ and a row vector $\bar\Phi$ of scalar multiplets. We wish to gauge the symmetry of unitary rotations

$$\Phi' = U\Phi, \qquad \bar\Phi' = \bar\Phi U^\dagger.$$

To do this, we introduce the covariant derivatives $\nabla_\alpha = D_\alpha + \mathcal{A}_\alpha$ and $\nabla_\mu = D_\mu + \mathcal{A}_\mu$. Gauge transformation acts on the superconnection in the standard manner,

$$\mathcal{A}'_\alpha = U\mathcal{A}_\alpha U^\dagger + UD_\alpha U^\dagger, \qquad \mathcal{A}'_\mu = U\mathcal{A}_\mu U^\dagger + UD_\mu U^\dagger.$$

The correct set of component fields is obtained by solving the Bianchi identity that arise from $\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta}$. The explicit solution reads

$$[\nabla_\alpha, \nabla_{\beta\gamma}] = i\epsilon_{\alpha\beta}\mathcal{W}_\gamma + i\epsilon_{\alpha\gamma}\mathcal{W}_\beta, \tag{A.4}$$

$$\nabla^\alpha\mathcal{W}_\alpha = 0, \qquad \nabla_\alpha\mathcal{W}_\beta = \nabla_{(\alpha}\mathcal{W}_{\beta)} = \mathcal{F}_{\alpha\beta}. \tag{A.5}$$

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Here $\mathcal{W}_\alpha$ is the gaugino superfield and $\mathcal{F}_{\alpha\beta}$ is the gauge field strength,

$$\mathcal{F}_{\alpha\beta} = -\frac{1}{2}\mathcal{F}_{\mu\nu}(\gamma^{\mu\nu}\epsilon)_{\alpha\beta}, \qquad \mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{A.6}$$

In Wess-Zumino gauge, these superfields take the form

$$\begin{aligned} \mathcal{A}_\alpha &= i\theta^\beta A_{\alpha\beta} + \theta^2\chi_\alpha, \\ \mathcal{A}_{\alpha\beta} &= A_{\alpha\beta} - i\theta_\alpha\chi_\beta - i\theta_\beta\chi_\alpha + \frac{i}{2}\theta^2 F_{\alpha\beta}, \\ \mathcal{W}_\alpha &= \chi_\alpha + \theta^\beta F_{\alpha\beta} - \frac{i}{2}\theta^2\nabla_\alpha{}^\beta\chi_\beta. \end{aligned} \tag{A.7}$$

General gauge theory. Let us spell out the full Lagrangian for $\mathcal{N}=1$ supersymmetric gauge theories. First, the gauge kinetic term consists of the Chern-Simons and Yang-Mills terms

$$\begin{aligned} \frac{k}{16\pi}\int d^2\theta \mathrm{Tr}\left(-iA\mathcal{W} + \frac{1}{6}\{A^\beta, A^\gamma\}A_{\beta\gamma}\right) &= \frac{k}{4\pi}\mathrm{Tr}\left[\varepsilon^{\mu\nu\rho}\left(A\partial A + \frac{2}{3}A^3\right)_{\mu\nu\rho} + i\chi\chi\right], \\ -\frac{1}{8g^2}\int d^2\theta \mathrm{Tr}\mathcal{W}^2 &= \frac{1}{4g^2}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu} - 2i\chi\nabla\!\!\!\!/\,\chi\right). \end{aligned} \tag{A.8}$$

The matter kinetic term and the superpotential term are given by

$$\begin{aligned} \frac{1}{4}\int d^2\theta \nabla\bar{\Phi}\nabla\Phi &= -\nabla^\mu\bar{\phi}\nabla_\mu\phi + \bar{F}F + i\bar{\psi}\nabla\!\!\!\!/\,\psi - i\bar{\psi}\chi\phi + i\bar{\phi}\chi\psi, \\ \frac{i}{2}\int d^2\theta W(\Phi) &= -\frac{i}{2}W_{ij}\psi^i\psi^j - W_i F^i. \end{aligned} \tag{A.9}$$

Here the matter superfields are taken to be real in the second line. We have also slightly redefined the matter component fields,

$$\phi = \Phi|, \qquad \psi_\alpha = -i\nabla_\alpha\Phi|, \qquad F = -\frac{i}{2}\nabla^\alpha\nabla_\alpha\Phi|. \tag{A.10}$$

The supersymmetry transformation rules for the component fields are

$$\delta\phi = i\xi\psi, \qquad \delta\psi = -\nabla\!\!\!\!/\,\phi\xi - F\xi, \qquad \delta F = i\xi\nabla\!\!\!\!/\,\psi - i\xi\chi\phi. \tag{A.11}$$

$$\delta_\xi A_\mu = -i\xi\gamma_\mu\chi, \qquad \delta_\xi\chi = -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\xi. \tag{A.12}$$

References

[1] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[2] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955].

[3] J. Bagger and N. Lambert, *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[4] A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260.

[5] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[6] D. Gaiotto and E. Witten, *Janus Configurations, Chern-Simons couplings, and the theta-angle in $N = 4$ super Yang-Mills theory*, arXiv:0804.2907.

[7] H.-C. Kao and K.-M. Lee, *Selfdual Chern-Simons systems with an $N = 3$ extended supersymmetry*, *Phys. Rev.* **D 46** (1992) 4691 [hep-th/9205115].

[8] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-matter theories*, *JHEP* **08** (2007) 056 [arXiv:0704.3740].

[9] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[10] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *$N = 8$ superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242].

[11] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple M2-branes*, *JHEP* **05** (2008) 105 [arXiv:0803.3803].

[12] N. Lambert and D. Tong, *Membranes on an orbifold*, arXiv:0804.1114.

[13] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256].

[14] P.M. Ho and Y. Matsuo, *A toy model of open membrane field theory in constant 3-form flux*, *Gen. Rel. Grav.* **39** (2007) 913 [hep-th/0701130].

[15] D.S. Berman, *M-theory branes and their interactions*, *Phys. Rept.* **456** (2008) 89 [arXiv:0710.1707].

[16] S. Mukhi and C. Papageorgakis, *M2 to D2*, *JHEP* **05** (2008) 085 [arXiv:0803.3218].

[17] D.S. Berman, L.C. Tadrowski and D.C. Thompson, *Aspects of multiple membranes*, arXiv:0803.3611.

[18] A. Morozov, *On the problem of multiple M2 branes*, *JHEP* **05** (2008) 076 [arXiv:0804.0913].

[19] U. Gran, B.E.W. Nilsson and C. Petersson, *On relating multiple M2 and D2-branes*, arXiv:0804.1784.

[20] P.-M. Ho, R.-C. Hou and Y. Matsuo, *Lie 3-algebra and multiple M2-branes*, *JHEP* **06** (2008) 020 [arXiv:0804.2110].

[21] J. Gomis, A.J. Salim and F. Passerini, *Matrix theory of type IIB plane wave from membranes*, arXiv:0804.2186.

[22] E.A. Bergshoeff, M. de Roo and O. Hohm, *Multiple M2-branes and the embedding tensor*, *Class. and Quant. Grav.* **25** (2008) 142001 [arXiv:0804.2201].

[23] K. Hosomichi, K.-M. Lee and S. Lee, *Mass-deformed Bagger-Lambert theory and its BPS objects*, arXiv:0804.2519.

[24] G. Papadopoulos, *M2-branes, 3-Lie algebras and Plücker relations*, *JHEP* **05** (2008) 054 [arXiv:0804.2662].

[25] J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, arXiv:0804.3078.

JHEP07(2008)091

[26] H. Shimada, *β-deformation for matrix model of M-theory*, `arXiv:0804.3236`.

[27] G. Papadopoulos, *On the structure of k-Lie algebras*, *Class. and Quant. Grav.* **25** (2008) 142002 [`arXiv:0804.3567`].

[28] P.-M. Ho and Y. Matsuo, *$M5$ from $M2$*, `arXiv:0804.3629`.

[29] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [`arXiv:0805.1012`].

[30] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, *$N = 8$ superconformal gauge theories and $M2$ branes*, `arXiv:0805.1087`.

[31] P.-M. Ho, Y. Imamura and Y. Matsuo, *$M2$ to $D2$ revisited*, `arXiv:0805.1202`.

[32] A. Morozov, *From simplified BLG action to the first-quantized M-theory*, `arXiv:0805.1703`.

[33] Y. Honma, S. Iso, Y. Sumitomo and S. Zhang, *Janus field theories from multiple $M2$ branes*, `arXiv:0805.1895`.

[34] H. Fuji, S. Terashima and M. Yamazaki, *A new $N = 4$ membrane action via orbifold*, `arXiv:0805.1997`.

[35] P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, *$M5$-brane in three-form flux and multiple $M2$-branes*, `arXiv:0805.2898`.

[36] C. Krishnan and C. Maccaferri, *Membranes on calibrations*, `arXiv:0805.3125`.

[37] Y. Song, *Mass deformation of the multiple $M2$ branes theory*, `arXiv:0805.3193`.

[38] I. Jeon, J. Kim, N. Kim, S.-W. Kim and J.-H. Park, *Classification of the BPS states in Bagger-Lambert theory*, `arXiv:0805.3236`.

[39] M. Li and T. Wang, *$M2$-branes coupled to antisymmetric fluxes*, `arXiv:0805.3427`.

[40] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions I: local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [`arXiv:0705.0022`].

[41] S. Lee, *Superconformal field theories from crystal lattices*, *Phys. Rev.* **D 75** (2007) 101901 [`hep-th/0610204`].

[42] S. Lee, S. Lee and J. Park, *Toric AdS_4/CFT_3 duals and M-theory crystals*, *JHEP* **05** (2007) 004 [`hep-th/0702120`].

[43] S. Kim, S. Lee, S. Lee and J. Park, *Abelian gauge theory on $M2$-brane and toric duality*, *Nucl. Phys.* **B 797** (2008) 340 [`arXiv:0705.3540`].

[44] P.B. Kronheimer, *The construction of ALE spaces as hyperKähler quotients*, *J. Diff. Geom.* **29** (1989) 665.

[45] S. Franco, A. Hanany, K.D. Kennaway, D. Vegh and B. Wecht, *Brane dimers and quiver gauge theories*, *JHEP* **01** (2006) 096 [`hep-th/0504110`].

[46] S. Franco et al., *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [`hep-th/0505211`].

[47] A. Hanany and D. Vegh, *Quivers, tilings, branes and rhombi*, *JHEP* **10** (2007) 029 [`hep-th/0511063`].

[48] M. Porrati and A. Zaffaroni, *M-theory origin of mirror symmetry in three dimensional gauge theories*, *Nucl. Phys.* **B 490** (1997) 107 [`hep-th/9611201`].

JHEP07(2008)091

[49] R. Brooks and S.J. Gates Jr., *Extended supersymmetry and super-BF gauge theories*, *Nucl. Phys.* **B 432** (1994) 205 [hep-th/9407147].

[50] A. Kapustin and M.J. Strassler, *On mirror symmetry in three dimensional Abelian gauge theories*, *JHEP* **04** (1999) 021 [hep-th/9902033].